Exam Review

(4) $ma = -kv$

$m\frac{dv}{dt} = -kv$ Separable

$\frac{dv}{v} = -\frac{k}{m}dt$

$\int \frac{dv}{v} = \int -\frac{k}{m}dt$

$\ln|v| = -\frac{k}{m}t + C_1$

$e^{LS} = e^{RS}$: $|v| = e^{-\frac{k}{m}t + C_1}$

$v = \pm \underbrace{e^{C_1}}_{C} \cdot e^{-\frac{k}{m}t}$

$v = Ce^{-\frac{k}{m}t}$

$v = v_0$, $t = 0$: $v_0 = C$

$v = v_0 e^{-\frac{k}{m}t}$

(5) $x^2y'' - xy' + y = x^3$

1) y_c

Cauchy-Euler DE $m(m-1) - m + 1 = 0$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_1 = x \quad , \quad y_2 = x \ln x$$

$$y_c = C_1 x + C_2 x \ln x$$

2) $$W = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x(1+\ln x) - x\ln x = x$$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ f(x) & \sim \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x\ln x \\ x & \sim \end{vmatrix}$$

$$= -x^2 \ln x$$

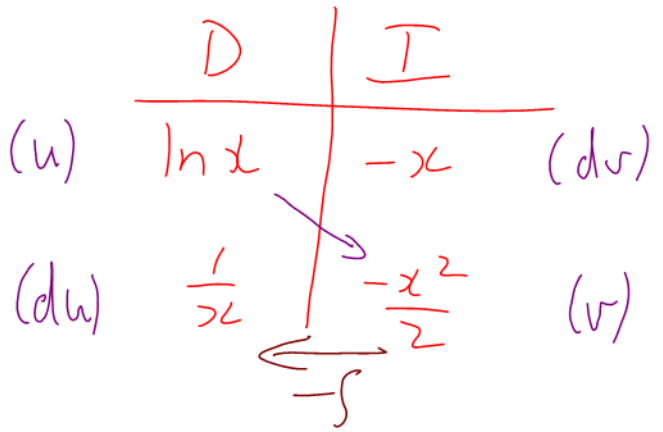
$$W_2 = \begin{vmatrix} x & 0 \\ 1 & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix}$$

$$= x^2$$

3) $$u_1' = \frac{W_1}{W} = -x\ln x$$

Caution:
Standard form
$y'' - \ldots = \underbrace{x}_{f(x)}$



$$u_1 = \int -x \ln x \, dx$$

$$= uv - \int v \, du$$

$$= -\frac{x^2}{2} \ln x + \int \frac{x}{2} dx$$

$$= -\frac{x^2}{2} \ln x + \frac{x^2}{4}$$

4) $$u_2' = \frac{W_2}{W} = x$$

$$u_2 = \int x \, dx = \frac{x^2}{2}$$

5) $$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{x^2}{2} \ln x + \frac{x^2}{4}\right) x + \frac{x^2}{2} (x \ln x)$$

$$= \frac{x^3}{4}$$

6) y

$$y = y_c + y_p$$

$$y = C_1 x + C_2 x \ln x + \frac{x^3}{4}$$

(6) Find a second solution.

$$x^3 y'' - x y' + y = 0$$

$$y'' - \frac{1}{x^2} y' + \frac{y}{x^3} = 0 \quad \text{Standard Form}$$

$$P(x) = -x^{-2}$$

$$-\int P(x)\,dx = \int x^{-2}\,dx$$

$$= -x^{-1}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= x \int \frac{e^{-x^{-1}}}{x^2}\,dx$$

Sub $u = -x^{-1}$

$du = \frac{1}{x^2}\,dx$

$$= x \int e^u\,du$$

$$= x\,[e^u]$$

$$= x\,e^{-x^{-1}}$$

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 x + C_2 x e^{-x^{-1}}$$

(7) $y''' - y'' = 2 + 2\cos x$

1) y_c

Constant Coefficient DE

$$m^3 - m^2 = 0$$

$$m^2(m-1) = 0$$

$$m = 0, 0, 1$$

$$y_1 = e^{0x} = 1, \quad y_2 = x, \quad y_3 = e^x$$

$$y_c = C_1 + C_2 x + C_3 e^x$$

2) y_p

$$\begin{cases} f(x) = 2 + 2\cos x \\ f'(x) = -2\sin x \\ \vdots \end{cases}$$

$y_p = A + B\cos x + C\sin x$?

duplication

$y_p = Ax^2 + B\cos x + C\sin x$

3) $y_p \to DE$ (find A, B, C)

$$y_p = Ax^2 + B\cos x + C\sin x$$
$$y_p' = 2Ax - B\sin x + C\cos x$$
$$y_p'' = 2A - B\cos x - C\sin x$$
$$y_p''' = B\sin x - C\cos x$$

$$y''' - y'' = 2 + 2\cos x$$

$$B\sin x - C\cos x - (2A - B\cos x - C\sin x) = 2 + 2\cos x$$

Constant: $-2A = 2 \Rightarrow A = -1$

$\cos x$: $B - C = 2$ (1)

$\sin x$: $B + C = 0$ (2)

(1)+(2): $2B = 2 \Rightarrow B = 1$

$B = 1 \to$ (2): $C = -1$

$$y_p = -x^2 + \cos x - \sin x$$

4) y

$$y = y_c + y_p$$
$$y = C_1 + C_2 x + C_3 e^x - x^2 + \cos x - \sin x$$

(8) a) $mx'' + \beta x' + kx = 0$

$m = 1kg$ $\quad \beta = 0$ $\quad k = 9 \frac{N}{m}$

$$x'' + 9x = 0$$

Constant Coefficient DE

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm\sqrt{-9}$$

$$m = \pm 3i \qquad (\alpha = 0, \beta = 3)$$

~~$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$~~

$$x = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

$$\boxed{x = C_1 \cos 3t + C_2 \sin 3t}$$

$\ominus$ $\quad t = 0 \quad : \quad 2 = C_1$

$\oplus$ $\quad x = 2$

$$\boxed{x = 2\cos 3t + C_2 \sin 3t}$$

$$x' = -6\sin 3t + 3C_2 \cos 3t$$

$t=0$

$x'=-1$ $\therefore$ $-1 = 3C_2$

$C_2 = -\frac{1}{3}$

$$x = 2\cos 3t - \frac{1}{3}\sin 3t$$