8.2 Solving Homogeneous Systems

Case 3: Repeated λ without enough eigenvectors

Ex: Solve $\vec{X}' = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix} \vec{X}$

$\lambda = 3, 3, 3$

$\vdots$

$\vec{K}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{P} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{Q}$: $[A - \lambda I]\vec{Q} = \vec{P}$

$[A - 3I \mid \vec{P}]$

$q_1 \; q_2 \; q_3$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ RREF

$q_3 = a$

$q_1 = 1$

$q_2 = -2$

$\vec{Q} = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}$

Choose any nonzero $\vec{Q}$: $\vec{Q} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

Now $\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2 + C_3\vec{X}_3$

where $\vec{X}_1 = \vec{K}_1 e^{\lambda t}$

$\vec{X}_2 = (\vec{K}_1 t + \vec{P})e^{\lambda t}$

$\vec{X}_3 = (\vec{K}_1 \frac{t^2}{2} + \vec{P}t + \vec{Q})e^{\lambda t}$

$$\vec{X} = C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t} + C_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{3t} + C_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right) e^{3t}$$

Case 4: $\lambda = \alpha \pm \beta i$

Find an eigenvector $\vec{K}_1$ for $\lambda = \alpha + \beta i$.

Let $\vec{B}_1 = Re(\vec{K}_1)$ and $\vec{B}_2 = Im(\vec{K}_1)$.

Then $\vec{X}_1 = e^{\alpha t}[\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t]$

$\vec{X}_2 = e^{\alpha t}[\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t]$.

Quick ex:

$\vec{K}_1 = \begin{bmatrix} 2i \\ 5-7i \end{bmatrix}$

Find $\vec{B}_1$, $\vec{B}_2$.

$\vec{K}_1 = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + i \begin{bmatrix} 2 \\ -7 \end{bmatrix}$ ($\vec{B}_1$, $\vec{B}_2$)

$\vec{B}_1 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\vec{B}_2 = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$

Ex: Solve $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} \vec{x}$.

λ: $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 + 4 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{-16}}{2}$$

$$\lambda = \frac{6 \pm 4i}{2}$$

$$\lambda = 3 \pm 2i \qquad (\alpha = 3, \beta = 2)$$

$\lambda = 3 + 2i$: $[A - \lambda I \mid \vec{0}]$

$[A - (3+2i) I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

$\frac{R_2}{2}$ $\begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$ RREF

(x_1, x_2)

Note: Nontrivial solutions must exist.

$\uparrow$

$x_2 = a$

$x_1 - i x_2 = 0 \implies x_1 = ai$

$\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix} a$

Choose any nonzero eigenvector: $\vec{K}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\vec{K}_1 = \begin{bmatrix} \quad \end{bmatrix} + i \begin{bmatrix} \quad \end{bmatrix}$

$\vec{B}_1 = Re(\vec{K}_1)$ $\quad$ $\vec{B}_2 = Im(\vec{K}_1)$

$\vec{K}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{B}_1$ $\quad$ $\vec{B}_2$

$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$ where

$\vec{X}_1 = e^{\alpha t} [\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t]$

$\vec{X}_2 = e^{\alpha t} [\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t]$ $\quad (\alpha = 3, \beta = 2)$

$\vec{X} = C_1 e^{3t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 2t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 2t \right) + C_2 e^{3t} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos 2t \right)$

Ex: Solve $\vec{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 8 \\ 0 & -1 & -1 \end{bmatrix} \vec{x}$

λ: $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 8 \\ 0 & -1 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 8 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(-1-\lambda) + 8\right] = 0$$

$$(2-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$\lambda = 2$ $\qquad$ $\lambda = \dfrac{2 \pm \sqrt{-16}}{2}$

$$\lambda = \frac{2 \pm 4i}{2}$$

$$\lambda = 1 \pm 2i$$

$\lambda = 2$: $\qquad [A - \lambda I \mid \vec{0}]$

$$[A - 2I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

$x_1 = a$

$x_2 = 0$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

Choose any nonzero eigenvector: $\vec{K}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

$\lambda = 1+2i$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A - (1+2i) I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 1-2i & 0 & 0 & 0 \\ 0 & 2-2i & 8 & 0 \\ 0 & -1 & -2-2i & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$\frac{R_1}{1-2i}$, $\frac{R_3}{-1}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2+2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note: Nontrivial solutions exist.

$$\uparrow$$

$$x_3 = a$$

$$x_1 = 0$$

$$x_2 + (2+2i)x_3 = 0 \Rightarrow x_2 = -(2+2i)a$$

$$\vec{x} = \begin{bmatrix} 0 \\ -2-2i \\ 1 \end{bmatrix} a$$

Choose any nonzero $\vec{x}$: $\quad \vec{K} = \begin{bmatrix} 0 \\ 2+2i \\ -1 \end{bmatrix}$

To Be Continued

$$\vec{B}_1 = Re(\vec{K}),\ \vec{B}_2 = Im(\vec{K})$$

$\vec{X}_2 =$

$\vec{X}_3 =$