8.2 Solving Homogeneous Systems Cont'd

Case 2: Repeated λ with enough Eigenvectors

$$\vec{X} = \sum c_i \vec{K}_i e^{\lambda_i t}$$

(same formula as Case 1)

Ex: Solve $\vec{X}' = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{X}$

Fact
The eigenvalues of an upper triangular, lower triangular, or diagonal matrix are the diagonal entries.

$$\lambda = 1, 2, 2$$

$\lambda = 1$: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ RREF

$\uparrow$

$x_3 = a$

$x_1 + 3x_3 = 0 \Rightarrow x_1 = -3a$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} a$$

Choose any nonzero eigenvector:

$$\vec{K}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_1 = a$, $x_2 = b$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} b$$

Use basis vectors:

$$\vec{K}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{K}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

$$\vec{X} = C_1 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Case 3: Repeated λ without enough eigenvectors

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t}$$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{P}) e^{\lambda t} \quad \text{where } [A-\lambda I]\vec{P} = \vec{K}_1$$

$$\vec{X}_3 = (\vec{K}_1 \frac{t^2}{2} + \vec{P}t + \vec{Q}) e^{\lambda t} \quad \text{where } [A-\lambda I]\vec{Q} = \vec{P}$$

If λ is repeated twice, use $\vec{X}_1$ and $\vec{X}_2$.
If λ is repeated three times, use $\vec{X}_1$, $\vec{X}_2$ and $\vec{X}_3$.

Ex: Solve $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\vec{x}$

given $\lambda = 1, 1$.

$\lambda = 1$: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$\uparrow$
$x_2 = a$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2a$

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} a$$

Choose any nonzero eigenvector: $\vec{K}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{P}$: $[A - \lambda I]\vec{P} = \vec{K}_1$

$[A - I \mid \vec{K}_1]$

$$\begin{matrix} P_1 & P_2 \end{matrix}$$
$$\left[\begin{matrix} 2 & -4 \\ 1 & -2 \end{matrix} \middle| \begin{matrix} 2 \\ 1 \end{matrix}\right]$$

$$\left[\begin{matrix} 1 & -2 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix}\right] \text{ RREF}$$

$\uparrow$

$P_2 = a$

$P_1 - 2P_2 = 1 \Rightarrow P_1 = 1 + 2a$

$$\vec{P} = \begin{bmatrix} 1+2a \\ a \end{bmatrix}$$

Choose any nonzero $\vec{P}$: $\vec{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{X}_1 = \vec{K}_1 e^{\lambda t} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t$$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{P}) e^{\lambda t} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^t$$

$$\vec{X} = C_1 \vec{X}_1 + C_2 \vec{X}_2$$

$$\vec{X} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + C_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^t$$

Ex: Solve $\vec{x}' = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix} \vec{x}$

A is lower triangular $\Rightarrow \lambda = 3, 3, 3$

$\lambda = 3$: $[A - 3I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$\uparrow$

$x_3 = a$

$x_1 = 0, \quad x_2 = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a$$

Choose any nonzero eigenvector: $\vec{K_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{P}$: $[A - \lambda I]\vec{P} = \vec{K_1}$

$[A - 3I \mid \vec{K_1}]$

$P_1 \quad P_2 \quad P_3$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$\uparrow$

$P_3 = a$

$$p_1 = 0$$
$$p_2 = 1$$

$$\vec{p} = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}$$

Choose any nonzero $\vec{p} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vec{Q}$: $[A - \lambda I]\vec{Q} = \vec{p}$

To Be Continued