8.2 Solving Homogeneous Systems

Solve $\vec{X}' = A\vec{X}$

4 cases: 1) A has distinct real eigenvalues
2) A has repeated real λ with enough eigenvectors
3) A has repeated real λ with geometric multiplicity = 1
4) A has complex λ

Case 1: Distinct Real Eigenvalues

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

where λ_i = eigenvalues of A

$\vec{K}_i$ = eigenvectors of A

Ex: Solve $\vec{X}' = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \vec{X}$

Find λ:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

$\lambda = 1$: $\quad [A - \lambda I \mid \vec{0}]$

$$[A - I \mid \vec{0}]$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 \end{matrix}$$

$$\begin{bmatrix} \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 + 2R_1 \quad \text{RREF}$$

$$\uparrow$$

$$x_2 = a$$

Note: Avoid using t as the parameter.

$$x_1 - x_2 = 0 \rightarrow x_1 = x_2 \rightarrow x_1 = a$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} a$$

Choose any nonzero eigenvector:

$$\vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 4$: $\quad [A - 4I \mid \vec{0}]$

$$\begin{bmatrix} -2 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

$\frac{R_1}{-2}$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

$x_1 \quad x_2$

R_2+2R_1

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ RREF

$x_2 = a$

$x_1 + \frac{1}{2}x_2 = 0 \rightarrow x_1 = -\frac{x_2}{2} \rightarrow x_1 = -\frac{a}{2}$

$$\vec{x} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} a$$

Choose any nonzero eigenvector

$$\vec{K}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ etc.}$$

$$\vec{X} = C_1 \vec{K}_1 e^{\lambda_1 t} + C_2 \vec{K}_2 e^{\lambda_2 t}$$

$$\vec{X} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{4t}$$

RECALL

To find eigenvalues : $|A-\lambda I| = 0$

To find eigenvectors : $[A-\lambda I \mid \vec{0}]$

Ex: Solve $\vec{X}' = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{X}$

λ: $|A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Cofactor Expansion

Checkerboard Pattern for Signs $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$(1-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)^2 - 4\right] = 0$$

$$(1-\lambda)(\lambda^2 - 8\lambda + 12) = 0$$

$$(1-\lambda)(\lambda-2)(\lambda-6) = 0$$

$$\lambda = 1, 2, 6$$

$\lambda = 1$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A - I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{R_1}{3}$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$3 - \frac{4}{3} = \frac{9}{3} - \frac{4}{3}$

$\frac{3}{5}R_2$

$$\left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + \frac{2}{3}R_2$

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

RREF

x_3 is free

$x_1 = 0$

$x_2 = 0$

$x_3 = a$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} a$$

$$\vec{k}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$:

$$[A - 2I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$x_2 = a$

$x_1 - x_2 = 0 \rightarrow x_1 = x_2 \rightarrow x_1 = a$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} a$$

$$\vec{K}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 6$:

$$[A - 6I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$$\rightsquigarrow \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$x_2 = a$

$x_1 + x_2 = 0 \rightarrow x_1 = -x_2 \rightarrow x_1 = -a$

$x_3 = 0$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} a$$

$$\vec{K}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ etc.}$$

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

$$\vec{X} = C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{6t}$$

Case 2 : Repeated λ with enough eigenvectors

$$\vec{X} = \sum C_i \vec{K}_i e^{\lambda_i t}$$

(Same formula as Case 1)