Omit Section 8.3

Asst 4

Due Tues April 7, 8:30am

Final Exam

Tues April 14
8:30 - 11:30 am
CHW 233

Final Exam: 10 Questions

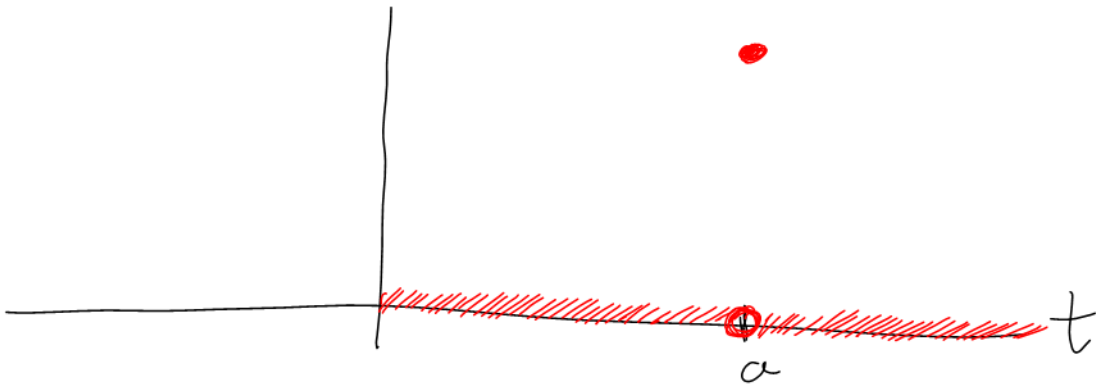
Chapter	# of Questions
2	2
3	1
4	2
5	1
6	1
7	2
8	1

7.5 The Dirac Delta

Let $a \geq 0$.

The Dirac Delta is

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}$$



$\delta(t-a)$ is a "generalized function" or "distribution"

Models: point masses
point charges

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

Ex: 1 kg mass at rest at equilibrium position. No damping.
Spring constant = 1 $\frac{N}{m}$.
Mass is struck at $t=\pi$ seconds, corresponding to an external force of $f(t) = \delta(t-\pi)$
Find $x(t)$ using $\mathcal{L}$.

Spring-Mass System

$$mx'' + \beta x' + kx = f(t)$$

$$1x'' + 1x = \delta(t-\pi)$$

$$x'' + x = \delta(t-\pi), \quad x'(0)=0, \quad x(0)=0$$

1) Apply $\mathcal{L}$

$$s^2X(s) - sx(0) - x'(0) + X(s) = e^{-\pi s}$$

$x(0)=0$
$x'(0)=0$ $\Rightarrow$ $s^2X(s) + X(s) = e^{-\pi s}$

2) Solve for $X(s)$

$$(s^2+1)X(s) = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2+1}$$

3) Apply $\mathcal{L}^{-1}$

$$x(t) = f(t-\pi)\,u(t-\pi)$$

$$\begin{cases} F(s) = \dfrac{1}{s^2+1} \\ f(t) = \sin t \end{cases}$$

$$x(t) = \sin(t-\pi)\,u(t-\pi)$$

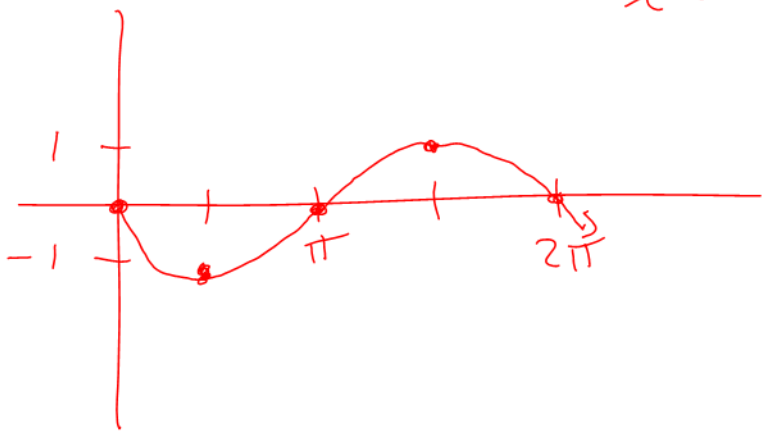
Alternatively:

$$\sin(t-\pi) = \sin t \cos\pi - \cos t \sin\pi$$
$$= -\sin t$$

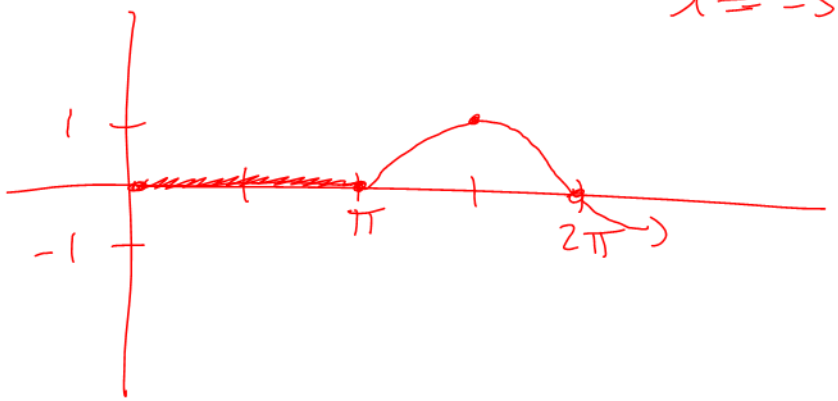
$$x(t) = (-\sin t)\,u(t-\pi)$$

To visualize:

$x = -\sin t$



$x = -\sin t\, u(t-\pi)$



8.1 Linear Systems of DEs

Ex:
$$\begin{cases} \frac{dx}{dt} = x - 3y + t \\ \frac{dy}{dt} = x - y + 1 \end{cases}$$

In matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Symbolically:

$$\vec{X}' = A\vec{X} + \vec{F}$$

System is homogeneous if $\vec{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution to the system:

$$\vec{X} = \begin{bmatrix} \quad \end{bmatrix} \begin{matrix} \leftarrow x \text{ (as a function of } t) \\ \leftarrow y \text{ (as a function of } t) \end{matrix}$$

Ex: Check that $\vec{X} = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$ solves $\vec{X}' = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\vec{X}$. solution

System of DEs

LS of system $= \vec{X}'$

$= \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$

RS of system $= \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\vec{X}$

$= \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$

$= \begin{bmatrix} 2e^{4t} + 2e^{4t} \\ -2e^{4t} - 6e^{4t} \end{bmatrix}$

$= \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$ ✓

Preview

Notice $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

eigenvalue of A

eigenvector of A

Ex: Given general solution

$$\vec{x} = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix},$$

solve if $\vec{x}(0) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

$t=0$:

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$c_1 + c_2 = 6 \quad (1)$$

$$c_1 - 2c_2 = 3 \quad (2)$$

(1) − (2):

$$3c_2 = 3$$

$$c_2 = 1$$

$c_2 = 1 \rightarrow$ (1): $c_1 = 5$

$$\vec{x} = 5 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$$