Test Review

Ex: Solve

b) $x^2y'' - xy' + 2y = 0$

Cauchy-Euler

$$m(m-1) - m + 2 = 0$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{-4}}{2} \qquad \sqrt{4}\sqrt{-1} = 2i$$

$$m = \frac{2 \pm 2i}{2}$$

$$m = 1 \pm i \qquad (\alpha = 1, \beta = 1)$$

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

$$y = x\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right]$$

Ex: Mass of 2kg is attached to a spring with spring constant $4 \frac{N}{m}$. There is a damping force of magnitude 6 times velocity. Find $x(t)$. Your answer will involve C_1 and C_2.

$$mx'' + \beta x' + kx = 0$$

$$2x'' + 6x' + 4x = 0$$

Constant Coefficient DE

$$2m^2 + 6m + 4 = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

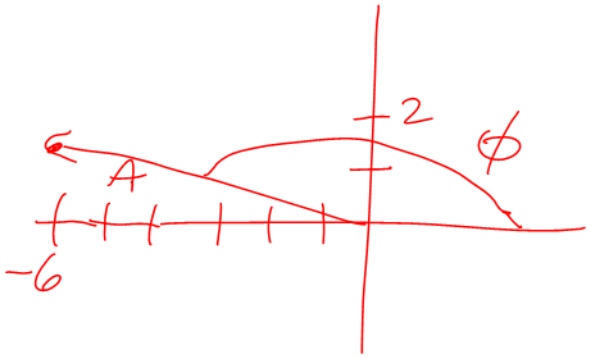
$$x(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Ex: Find the 2nd time when the mass passes through equilibrium position:

$x = 0$

$$x = -6\sin\frac{t}{2} + \sqrt{2}\cos\frac{t}{2}$$

$A\cos\phi = -6$ $\quad$ $A\sin\phi = \sqrt{2}$



$$A = \sqrt{(-6)^2 + \sqrt{2}^2} = \sqrt{38}$$

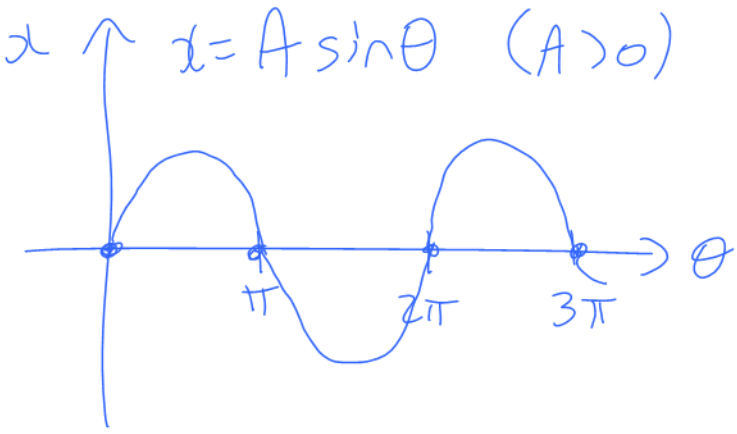
$$\phi = \tan^{-1}\left(\frac{\sqrt{2}}{-6}\right) (+\pi\,?)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{2}}{-6}\right) + \pi$$

$$\phi \approx 2.91 \text{ rads}$$

$$x = A\sin(\omega t + \phi)$$

$$x = \sqrt{38}\sin\left(\frac{t}{2} + 2.91\right)$$



$$\frac{t}{2} + 2.91 = 0, \pi, 2\pi, \ldots$$

$\frac{t}{2} + 2.91 = 0$ nonsense

$\frac{t}{2} + 2.91 = \pi \Rightarrow t \approx 0.46$ s (1st time)

$\frac{t}{2} + 2.91 = 2\pi$ (2nd time)

$\Rightarrow t \approx 7$ s

Ex: Use $y = \sum_{n=0}^{\infty} c_n x^n$ to find

$c_2, c_3, c_4, c_5, c_6, c_7$ in terms of c_0 and c_1.

$$y'' - 2x^2 y' = 0$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

DE: $$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^{n+1} = 0$$

$k = n-2$ $\qquad$ $k = n+1$

$n = k+2$ $\qquad$ $n = k-1$

$n=2 \Rightarrow k=0$ $\qquad$ $n=1 \Rightarrow k=2$

Start at the largest k-value ($k=2$)

1st two terms

$$2c_2 x^0 + 6c_3 x + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=2}^{\infty} 2(k-1) c_{k-1} x^k = 0$$

$$2C_2 + 6C_3 x + \sum_{k=2}^{\infty} \left[(k+2)(k+1)C_{k+2} - 2(k-1)C_{k-1}\right]x^k = 0$$

Set coefficients $= 0$

$$2C_2 = 0 \implies C_2 = 0$$
$$6C_3 = 0 \implies C_3 = 0$$

$$(k+2)(k+1)C_{k+2} - 2(k-1)C_{k-1} = 0$$

$$\implies C_{k+2} = \frac{2(k-1)C_{k-1}}{(k+2)(k+1)}, \quad k \geq 2$$

$(k=2)$ $C_4 = \dfrac{2 \cdot 1 \cdot C_1}{4 \cdot 3} = \dfrac{C_1}{6}$

$(k=3)$ $C_5 = \dfrac{2 \cdot 2 \cdot C_2}{5 \cdot 4} = 0$

$(k=4)$ $C_6 = \dfrac{2 \cdot 3 C_3}{6 \cdot 5} = 0$

$(k=5)$ $C_7 = \dfrac{2 \cdot 4 \cdot C_4}{7 \cdot 6} = \dfrac{8}{42} C_4 = \dfrac{8}{42}\left(\dfrac{C_1}{6}\right) = \dfrac{2C_1}{63}$

<u>Ex:</u> $x^2 y'' - 12xy' + 12y = x$

has $y_c = C_1 x + C_2 x^{12}$.

Solve using Variation of Parameters.

1) Get y_c

$y_1 = x$, $y_2 = x^{12}$

$$y_c = C_1 x + C_2 x^{12}$$

2) Get W, W_1, W_2

$$W = \begin{vmatrix} x & x^{12} \\ 1 & 12x^{11} \end{vmatrix} = 12x^{12} - x^{12} = 11x^{12}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x^{12} \\ \frac{1}{x} & \sim \end{vmatrix}$$

$$= -x^{11}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 \\ \sim & \frac{1}{x} \end{vmatrix}$$

$$= 1$$

standard form:

$$y'' - \frac{12}{x}y' + \frac{12}{x^2}y = \frac{1}{x}$$

$f(x)$

3) Get u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-x^{11}}{11x^{12}}$$

$$= \frac{-1}{11} x^{-1}$$

$$u_1 = \int \frac{-1}{11} x^{-1} \, dx$$

$$= \frac{-1}{11} \ln|x|$$

4) Get u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{11 x^{12}}$$

$$= \frac{1}{11} x^{-12}$$

$$u_2 = \int \frac{1}{11} x^{-12} \, dx$$

$$= \frac{-1}{121} x^{-11}$$

5) Get y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{-1}{11} x \ln|x| - \frac{1}{121} x^{-11} x^{12}$$

$$= \frac{-1}{11} x \ln|x| - \frac{1}{121} x$$

6) Get y

$$y = y_c + y_p$$

$$= C_1 x + C_2 x^{12} - \frac{1}{11} x \ln|x| - \frac{1}{121} x$$

$$= C_1 x + C_2 x^{12} - \frac{1}{11} x \ln|x|$$