7.4 Convolutions and Periodic Functions

Convolution of f and g:

$$f * g = \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

Sometimes written:

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Convolutions show up in:

- Vibrations
- Heat Transfer
- Fluid Mechanics
- Structural Analysis

FACT

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$$

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g$$

Ex: Find $e^t * \cos t$

Method I $\quad e^t * \cos t = \int_0^t e^\theta \cos(t-\theta)\, d\theta$

Long Integration by Parts (!!)

Method II

$$\mathcal{L}\{e^t * \cos t\} = F(s)\, G(s)$$

$f(t) = e^t$

$F(s) = \frac{1}{s-1}$

$g(t) = \cos t$

$G(s) = \frac{s}{s^2+1}$

$$\mathcal{L}\{e^t * \cos t\} = \frac{1}{s-1} \cdot \frac{s}{s^2+1}$$

$$e^t * \cos t = \mathcal{L}^{-1}\left\{\frac{1}{s-1} \cdot \frac{s}{s^2+1}\right\}$$

Partial Fractions

$$\frac{s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$s = A(s^2+1) + (Bs+C)(s-1)$$

$s=1$: $\quad 1 = 2A \implies A = \frac{1}{2}$

s^2 coefficient: $\quad 0 = A+B \implies B = -\frac{1}{2}$

Constant coefficient: $\quad 0 = A-C \implies C = \frac{1}{2}$

$$e^t * \cos t = \mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1}\right\}$$

$$= \frac{1}{2}e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+4)}\right\}$ using a convolution.

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^2+4} \cdot \frac{1}{s}\right\}$$

$$= f * g$$

$$\begin{cases} F(s) = \frac{2}{s^2+4} \\ f(t) = \sin 2t \\ G(s) = \frac{1}{s} \\ g(t) = 1 \end{cases}$$

$$= \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

$$= \int_0^t \sin 2\theta\, (1)\, d\theta$$

$$= \int_0^t \sin 2\theta\, d\theta$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^t$$

$$= -\frac{1}{2}\cos 2t + \frac{1}{2}$$

Ex: Solve for $f(t)$:

$$f(t) = 6t - \int_0^t f(\theta)\, e^{t-\theta}\, d\theta$$

Apply $\mathcal{L}$:

$$F(s) = \frac{6}{s^2} - \mathcal{L}\{f * e^t\}$$

$$F(s) = \frac{6}{s^2} - F(s) \cdot \frac{1}{s-1}$$

Solve for $F(s)$:

$$F(s) + F(s) \cdot \frac{1}{s-1} = \frac{6}{s^2}$$

$$\left[1 + \frac{1}{s-1} \right] F(s) = \frac{6}{s^2}$$

$$\frac{s}{s-1} F(s) = \frac{6}{s^2}$$

$$F(s) = \frac{6(s-1)}{s^3}$$

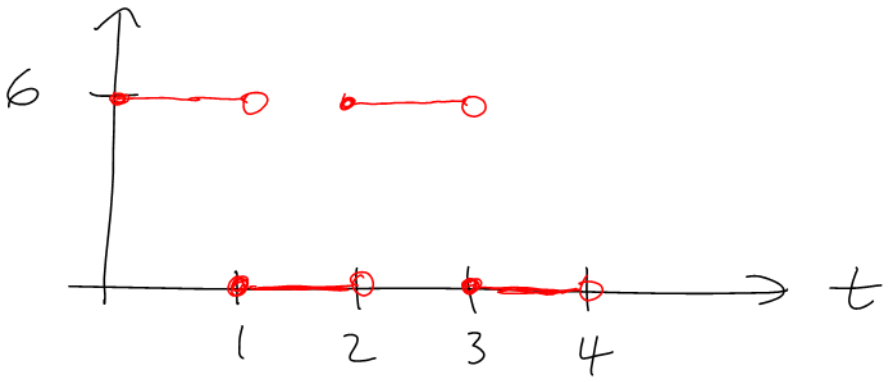
$-3\left(\frac{2}{s^3}\right)$

Apply $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{6}{s^2} - \frac{6}{s^3} \right\}$$

$$f(t) = 6t - 3t^2$$

Periodic Functions



$f(t)$ is periodic with period $T = 2$.

FACT

If $f(t)$ has period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Ex: Find $\mathcal{L}\{f(t)\}$ for $f(t)$ shown above. Simplify your answer.

$T = 2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 6e^{-st}\,dt + \int_1^2 0\,dt \right]$$

$$= \left(\frac{1}{1-e^{-2s}}\right) \frac{6e^{-st}}{-s} \Bigg|_{t=0}^{t=1}$$

$$= \frac{1}{1-e^{-2s}} \cdot \frac{6e^{-s} - 6}{-s}$$

$$= \frac{1}{1-(e^{-s})^2} \left(\frac{6}{s}\right)\left(\frac{e^{-s}-1}{-1}\right)$$

$$= \frac{1-e^{-s}}{1-(e^{-s})^2} \left(\frac{6}{s}\right)$$

$$= \frac{1-e^{-s}}{(1-e^{-s})(1+e^{-s})} \left(\frac{6}{s}\right)$$

$$= \frac{6}{2(1+e^{-2})}$$

Test Review

Ex: Solve

a) $x^2y'' - xy' + y = 0$

Cauchy-Euler DE

$$m(m-1) - m + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_1 = x \quad , \quad y_2 = x\ln x$$

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 x + C_2 x\ln x$$