7.3 Translations (Shifts) Cont'd

Recap: Shifting s

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+8s+25}\right\}$$

$s^2+8s+25 = (s+4)^2 + ?$

$= (s+4)^2 + 9$

$= (s+4)^2 + 3^2$

$2s = ?\,(s+4) + ?$

$= 2(s+4) + ?$

$= 2(s+4) - 8$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+4)-8}{(s+4)^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{2\cdot\frac{s+4}{(s+4)^2+3^2} - \frac{8}{3}\cdot\frac{3}{(s+4)^2+3^2}\right\}$$

$$= 2e^{-4t}\cos 3t - \frac{8}{3}e^{-4t}\sin 3t$$

Recap: Shifting t

Ex: Find $\mathcal{L}\{\sin 2t\, u(t-\pi)\}$

$= e^{-\pi s}\mathcal{L}\{f(t+\pi)\}$

$f(t) = \sin 2t$

$f(t+\pi) = \sin 2(t+\pi)$

$= \sin(2t+2\pi)$

$= -\sin 2t \cos 2\pi + \cos 2t \sin 2\pi$
$= \sin 2t$

$$= e^{-\pi s}\,\mathcal{L}\{\sin 2t\}$$

$$= \frac{2}{s^2+4}\,e^{-\pi s}$$

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^4}\right\}$

$$= f(t-3)\,u(t-3)$$

$F(s) = \dfrac{1}{s^4}$
$= \dfrac{1}{6}\cdot\dfrac{6}{s^4}$
$f(t) = \dfrac{1}{6}t^3$

$$= \frac{1}{6}(t-3)^3\,u(t-3)$$

Ex: Write using the Heaviside function:

$$f(t) = \begin{cases} 5, & 0 \le t < 3 \\ 7, & t \ge 3 \end{cases}$$

$$f(t) = ? + ?\,u(t-3)$$

$$f(t) = 5 + 2\,u(t-3)$$

Ex: Solve using the Laplace transform:

$$y' + 5y = f(t), \quad y(0) = 0$$

where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$

$f(t) = ? + ? \, u(t-3)$

$f(t) = 0 + 2\,u(t-3)$

$f(t) = 2\,u(t-3)$

DE: $y' + 5y = 2u(t-3)$, $y(0) = 0$

1) Apply $\mathcal{L}$

$sY(s) - y(0) + 5Y(s) = \mathcal{L}\{2u(t-3)\}$

$\mathcal{L}\{2u(t-3)\} = e^{-3s}\mathcal{L}\{f(t+3)\}$

$= e^{-3s}\mathcal{L}\{2\}$

$= \frac{2e^{-3s}}{s}$

$\begin{cases} f(t) = 2 \\ f(t+3) = 2 \end{cases}$

$sY(s) - y(0) + 5Y(s) = \frac{2e^{-3s}}{s}$

$y(0) = 0$: $sY(s) + 5Y(s) = \frac{2e^{-3s}}{s}$

2) Solve for $Y(s)$

$(s+5)Y(s) = \frac{2e^{-3s}}{s}$

$Y(s) = \frac{2e^{-3s}}{s(s+5)}$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2e^{-3s}}{s(s+5)} \right\} \quad (\star)$$

4) Partial Fractions

Caution: Do not include e^{-3s} in the partial fractions (!)

$$\frac{2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$2 = A(s+5) + Bs$$

$s=0$: $\quad 2 = 5A \implies A = \frac{2}{5}$

$s=-5$: $\quad 2 = -5B \implies B = -\frac{2}{5}$

5) Find $y(t)$

From $(\star)$ $\quad y(t) = \mathcal{L}^{-1}\left\{ \left(\frac{2}{5}\cdot\frac{1}{s} - \frac{2}{5}\cdot\frac{1}{s+5} \right) e^{-3s} \right\}$

$$= f(t-3)\,u(t-3)$$

$$F(s) = \frac{2}{5}\frac{1}{s} - \frac{2}{5}\frac{1}{s+5}$$

$$f(t) = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

$$f(t-3) = \frac{2}{5} - \frac{2}{5}e^{-5(t-3)}$$

$$= \left[\frac{2}{5} - \frac{2}{5}e^{-5(t-3)} \right] u(t-3)$$

Ex: Solve using $\mathcal{L}$:

$$x'' + 9x = \cos 3t,$$
$$x(0) = 0, \quad x'(0) = 1.$$

1) Apply $\mathcal{L}$

$$s^2X(s) - s\,x(0) - x'(0) + 9X(s) = \frac{s}{s^2+9}$$

$x(0)=0$, $x'(0)=1$:

$$s^2X(s) - 1 + 9X(s) = \frac{s}{s^2+9}$$

2) Solve for $X(s)$

$$s^2X(s) + 9X(s) = 1 + \frac{s}{s^2+9}$$

$$(s^2+9)X(s) = 1 + \frac{s}{s^2+9}$$

$$X(s) = \frac{1}{s^2+9} + \frac{s}{(s^2+9)^2}$$

3) Apply $\mathcal{L}^{-1}$:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{s^2+9} + \frac{1}{6}\frac{6s}{(s^2+9)^2}\right\}$$

$$x(t) = \frac{1}{3}\sin 3t + \frac{1}{6}t\sin 3t$$

7.4 Convolutions and Periodic Functions

Convolution of f and g:

$$f * g = \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

$$\mathcal{L}\{f * g\} = F(s)\, G(s)$$