7.2 Inverse Transforms Cont'd

FACT

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$$

Ex: Find:

a) $\mathcal{L}\{y(t)\}$

$= Y(s)$

b) $\mathcal{L}\{y'(t)\}$

$= s\,Y(s) - y(0)$

c) $\mathcal{L}\{y''(t)\}$

$= s^2 Y(s) - s\,y(0) - y'(0)$

Ex: Use the Laplace transform to solve $y'' - 5y' + 6y = 4e^t$, $y(0) = 3$, $y'(0) = 8$.

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{3s^2 - 10s + 11}{(s-1)(s-2)(s-3)} \right\}$$

(A)

4) Partial Fractions

$$\frac{3s^2-10s+11}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$

$$3s^2-10s+11 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

$s=1$: $\quad 4=2A \Rightarrow A=2$

$s=2$: $\quad 3=-B \Rightarrow B=-3$

$s=3$: $\quad 8=2C \Rightarrow C=4$

5) Find $y(t)$

From (A) $\quad y(t)=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{3}{s-2}+\frac{4}{s-3}\right\}$

$$y(t)=2e^{t}-3e^{2t}+4e^{3t}$$

7.3 Translations

$$\mathcal{L}\{e^{at}f(t)\}=F(s-a)$$

Ex: Evaluate

a) $\mathcal{L}\{e^{-3t}t^5\}$

$= F(s+3)$ $\qquad \begin{cases} f(t)=t^5 \\ F(s)=\frac{5!}{s^6} \end{cases}$

$= \frac{5!}{(s+3)^6}$

b) $\mathcal{L}\{e^{4t}\, t\sin 6t\}$

$= F(s-4)$

$\begin{cases} f(t) = t\sin 6t \\ F(s) = \dfrac{12s}{(s^2+36)^2} \end{cases}$

$= \dfrac{12(s-4)}{((s-4)^2+36)^2}$

FACT

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

Ex: $\mathcal{L}^{-1}\left\{\dfrac{-6}{s+2} + \dfrac{2}{(s+2)^2} + \dfrac{1}{(s+2)^3}\right\}$

$= \mathcal{L}^{-1}\{F(s+2)\}$

$= e^{-2t} f(t)$

$= e^{-2t}\left(-6+2t+\dfrac{t^2}{2}\right)$

$\begin{cases} F(s) = \dfrac{-6}{s} + \dfrac{2}{s^2} + \dfrac{1}{s^3} \\ = \dfrac{-6}{s} + 2\cdot\dfrac{1}{s^2} + \dfrac{1}{2}\cdot\dfrac{2}{s^3} \\ f(t) = -6 + 2t + \dfrac{1}{2}t^2 \end{cases}$

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s^2+6s+13}\right\}$

$s^2+6s+13$ doesn't factor (can't do partial fractions).
Complete the square.

$$\frac{6}{2} = 3$$

$$s^2 + 6s + 13 = (s+3)^2 + ?$$

$$= (s+3)^2 + 4$$

$$= (s+3)^2 + 2^2$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+1}{(s+3)^2+2^2}\right\}$$

Want $2s+1 = ?(s+3) + ?$

$$= 2(s+3) + ?$$

$$= 2(s+3) - 5$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+3)-5}{(s+3)^2+2^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$$

$$= 2e^{-3t}\cos 2t - \frac{5}{2}e^{-3t}\sin 2t$$

Ex: Solve using the Laplace transform

$$y'' + 4y' + 20y = 50e^t, \quad y(0) = 9, \quad y'(0) = -28.$$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + 4\left[sY(s) - y(0)\right] + 20Y(s) = \frac{50}{s-1}$$

$y(0) = 9$

$y'(0) = -28$

$$s^2Y(s) - 9s + 28 + 4sY(s) - 36 + 20Y(s) = \frac{50}{s-1}$$

2) Solve for $Y(s)$

$$\left[s^2 + 4s + 20\right]Y(s) = \frac{50}{s-1} + 9s + 8$$

$$Y(s) = \frac{50}{(s-1)(s^2+4s+20)} + \frac{9s+8}{s^2+4s+20}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{50}{(s-1)(s^2+4s+20)} + \frac{9s+8}{s^2+4s+20}\right\}$$

Combine into 1 fraction

$$y(t) = \mathcal{L}^{-1}\left\{\frac{50 + (9s+8)(s-1)}{(s-1)(s^2+4s+20)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{9s^2 - s + 42}{(s-1)(s^2+4s+20)} \right\} \quad (A)$$

4) Partial Fractions

$$\frac{9s^2 - s + 42}{(s-1)(s^2+4s+20)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+20}$$

↑ doesn't factor

$$9s^2 - s + 42 = A(s^2+4s+20) + (Bs+C)(s-1)$$

Can sub s-values and/or match coefficients

$s=1$: $\quad 50 = 25A \implies A = 2$

s^2 coefficient: $\quad 9 = A + B \implies B = 7$

constant: $\quad 42 = 20A - C$

$\implies 42 = 40 - C$

$\implies 2 = -C$

$\implies C = -2$

5) Find $y(t)$

From (A) $\quad y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s-1} + \frac{7s-2}{s^2+4s+20} \right\}$