7.2 Inverse Transforms Cont'd

Ex: Find $\mathcal{L}^{-1}\left\{\frac{12s^2+6}{s^4+s^2}\right\}$

Partial Fractions

$s^4+s^2 = s^2(s^2+1)$ ← quadratic factor

repeated linear factor

$$\frac{12s^2+6}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$12s^2+6 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

Sub any values and/or Match coefficients

$s=0$: $\boxed{6 = B}$

s^3 coefficient: $0 = A+C$ (1)

s^2 coefficient: $12 = B+D$

$\boxed{D=6}$

$s=1$: $18 = 2A + 2B + C + D$

$18 = 2A + 12 + C + 6$

$0 = 2A + C$ (2)

② − ①: $0 = A$

$A = 0 \rightarrow$ ①: $C = 0$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{s^2} + \frac{6}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{6 \cdot \frac{1}{s^2} + 6 \cdot \frac{1}{s^2+1}\right\}$$

$$= 6t + 6\sin t$$

FACT

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

FACT

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Ex: Use the Laplace transform to solve $\frac{dy}{dt} + 7y = 65\sin 4t$, $y(0) = 9$.

1) Apply $\mathcal{L}$

$$\mathcal{L}\{y'(t) + 7y\} = \mathcal{L}\{65\cos 4t\}$$

$$sY(s) - y(0) + 7Y(s) = \frac{65s}{s^2+16}$$

$y(0) = 9$: $$sY(s) - 9 + 7Y(s) = \frac{65s}{s^2+16}$$

2) Solve for $Y(s)$

$$sY(s) + 7Y(s) = 9 + \frac{65s}{s^2+16}$$

$$(s+7)Y(s) = 9 + \frac{65s}{s^2+16}$$

$$Y(s) = \frac{9}{s+7} + \frac{65s}{(s^2+16)(s+7)}$$

3) Apply $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{9}{s+7} + \frac{65s}{(s^2+16)(s+7)}\right\}$$

$$y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{65s}{(s^2+16)(s+7)}\right\}$$ (*)

4) Partial Fractions

$$\frac{65s}{(s^2+16)(s+7)} = \frac{A}{s+7} + \frac{Bs+C}{s^2+16}$$

$$65s = A(s^2+16) + (Bs+C)(s+7)$$

Sub any values and/or match coefficients.

$s=-7$: $-455 = A(65)$

$$\boxed{A=-7}$$

s^2 coefficient: $0 = A+B$

$$\boxed{B=7}$$

s coefficient: $65 = 7B+C$

$65 = 49+C$

$$\boxed{16=C}$$

5) Find $y(t)$

From ⋆

$$y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s+16}{s^2+16}\right\}$$

$$= 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s}{s^2+16} + 4\cdot\frac{4}{s^2+16}\right\}$$

$$= 9e^{-7t} - 7e^{-7t} + 7\cos 4t + 4\sin 4t$$

$$= 2e^{-7t} + 7\cos 4t + 4\sin 4t$$

Ex: Use the Laplace transform to solve

$$y'' - 5y' + 6y = 4e^t, \quad y(0)=3, \quad y'(0)=8.$$

1) Apply $\mathcal{L}$

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 5\left[sY(s) - y(0)\right] + 6Y(s) = \frac{4}{s-1}$$

$y(0)=3$
$y'(0)=8$:

$$s^2Y(s) - 3s - 8 - 5sY(s) + 15 + 6Y(s) = \frac{4}{s-1}$$

2) Solve for $Y(s)$

$$[s^2 - 5s + 6]Y(s) = 3s - 7 + \frac{4}{s-1}$$

$$Y(s) = \frac{3s-7}{s^2-5s+6} + \frac{4}{(s-1)(s^2-5s+6)}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3s-7}{(s-2)(s-3)} + \frac{4}{(s-1)(s-2)(s-3)}\right\}$$

CAUTION: Don't do partial fractions twice!

$$y(t) = \mathcal{L}^{-1}\left\{\frac{(3s-7)(s-1)+4}{(s-1)(s-2)(s-3)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{3s^2 - 10s + 11}{(s-1)(s-2)(s-3)} \right\}$$ ★

4) Partial Fractions