# Assignment 3

Due Thurs March 12, 8:30 am

## 7.1 The Laplace Transform Cont'd

$$\mathcal{L}\{\underbrace{f(t)}_{\text{function of } t}\} = \int_0^\infty e^{-st} f(t)\,dt = \underbrace{F(s)}_{\text{function of } s}$$

**Notation:** $F$ means the transformed version of $f$.

**Note:** $s$ is restricted to $s > 0$ so that $e^{-st} \to 0$, $te^{-st} \to 0$ as $t \to \infty$.

**Ex:** Find $\mathcal{L}\{e^{t+7}\}$

$$= \int_0^\infty e^{-st} e^{t+7}\,dt$$

$$= \int_0^\infty e^{t} \cdot e^{7} \cdot e^{-st}\,dt$$

$$= e^7 \int_0^\infty e^{(1-s)t}\,dt$$

$$= e^7 \left.\frac{e^{(1-s)t}}{1-s}\right|_{t=0}^{t=\infty}$$

$$= 0 - e^{7} \cdot \frac{1}{1-s}$$

Note:
$s > 1$ here
so that $e^{(1-s)t} \to 0$
as $t \to \infty$

$$= \frac{e^{7}}{s-1}$$

Ex: Find $\mathcal{L}\{\sin t\}$

$$= \int_0^\infty e^{-st} \sin t \, dt$$

$$\int u\,dv = uv - \int v\,du$$

| D | I |
|---|---|
| (u) $\sin t$ | $e^{-st}$ (dv) |
| $\cos t$ (du) | $-\frac{1}{s}e^{-st}$ (v) |

$$= -\frac{1}{s} e^{-st} \sin t \Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} \cos t \, dt$$

$$= \int_0^\infty \frac{1}{s} e^{-st} \cos t \, dt$$

| D | I |
|---|---|
| (u) $\cos t$ | $\frac{1}{s}e^{-st}$ (dv) |
| $-\sin t$ (du) | $-\frac{1}{s^2}e^{-st}$ (v) |

$$\int u\,dv = uv - \int v\,du$$

$$= -\frac{1}{s^2} e^{-st} \cos t \Big|_0^\infty - \int_0^\infty \frac{1}{s^2} e^{-st} \sin t \, dt$$

$$= \frac{1}{s^2} - \frac{1}{s^2} \int_0^\infty e^{-st} \sin t \, dt$$

$$= \frac{1}{s^2} - \frac{1}{s^2} \mathcal{L}\{\sin t\}$$

$$\Rightarrow \mathcal{L}\{\sin t\} = \frac{1}{s^2} - \frac{1}{s^2} \mathcal{L}\{\sin t\}$$

$$\mathcal{L}\{\sin t\} + \frac{1}{s^2} \mathcal{L}\{\sin t\} = \frac{1}{s^2}$$

$$\left(1 + \frac{1}{s^2}\right) \mathcal{L}\{\sin t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{\sin t\} = \frac{\frac{1}{s^2}}{1 + \frac{1}{s^2}} \qquad \left(\frac{s^2}{s^2}\right)$$

$$= \frac{1}{s^2 + 1}$$

We'll use the formula sheet from now on.

$$\mathcal{L}\{t\} = \mathcal{L}\{t^n\}_{n=1} = \frac{1!}{s^2} = \frac{1}{s^2}$$

$$\mathcal{L}\{\sin t\} = \mathcal{L}\{\sin \omega t\}_{\omega=1} = \frac{1}{s^2+1}$$

FACT

$\mathcal{L}$ is linear:

$$\mathcal{L}\{a f(t) + b g(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

Ex: $\mathcal{L}\{7t^2 - 3\cos 2t\}$

$$= 7\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right)$$

$$= \frac{14}{s^3} - \frac{3s}{s^2+4}$$

Ex: Fill in the table

| $f(t)$ | $\mathcal{L}\{f(t)\}$ |
|---|---|
| $3$ | $\frac{3}{s}$ |
| $7e^{2t}$ | $\frac{7}{s-2}$ |
| $4t^3$ | $4\left(\frac{3!}{s^4}\right) = \frac{24}{s^4}$ |
| $6\sin 5t$ | $6\left(\frac{5}{s^2+25}\right) = \frac{30}{s^2+25}$ |
| $-2\cos 8t$ | $-2\left(\frac{s}{s^2+64}\right) = \frac{-2s}{s^2+64}$ |

## 7.2 Inverse Transforms

$$\mathcal{L}\{t\} = \frac{1}{s^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$$

FACT

$\mathcal{L}^{-1}$ is linear:

$$\mathcal{L}^{-1}\{a F(s) + b G(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{3}{s^2+16} - \frac{1}{s^3}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{3}{4}\cdot\frac{4}{s^2+16} - \frac{1}{2}\cdot\frac{2}{s^3}\right\}$$

$$= \frac{3}{4}\sin 4t - \frac{1}{2}t^2$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{5s^2-5s+12}{s^3+s^2-12s}\right\}$

Partial Fractions

$$s^3+s^2-12s = s(s^2+s-12) = s(s+4)(s-3)$$

$$\frac{5s^2-5s+12}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-3}$$

$$5s^2-5s+12 = A(s+4)(s-3) + Bs(s-3) + Cs(s+4)$$

Can sub $s$-values.
Can match coefficients.

$s=0$: $12 = -12A \Rightarrow A = -1$

$s=3$: $42 = 21C \Rightarrow C = 2$

$s=-4$: $112 = 28B \Rightarrow B = 4$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{4}{s+4} + \frac{2}{s-3}\right\}$$

$$= -1 + 4e^{-4t} + 2e^{3t}$$