Solutions

(1) We need $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Let's calculate $\int_{-\infty}^{\infty} f(x)\,dx$:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} 0\,dx + k\int_{1}^{2} \frac{1}{x}\,dx + \int_{2}^{\infty} 0\,dx$$

$$= 0 + k\left[\ln|x|\right]_1^2 + 0$$

$$= k\left[\ln 2\right]$$

Now $k \cdot \ln 2 = 1$

$$\boxed{k = \frac{1}{\ln 2}}$$

At this point we can check that $f(x) \geq 0$ for all x-values. ✓

So $k = \frac{1}{\ln 2}$ makes $f(x)$ a valid probability density function.

(2) We need $\int_{-\infty}^{\infty} f(x)dx = 1.$

Let's calculate $\int_{-\infty}^{\infty} f(x)dx$:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0\,dx + k\int_{0}^{2} x^4 dx + \int_{2}^{\infty} 0\,dx$$

$$= 0 + k\left[\frac{x^5}{5}\right]_0^2 + 0.$$

$$= k\left(\frac{32}{5}\right)$$

Now $k\left(\frac{32}{5}\right) = 1$

$$\boxed{k = \frac{5}{32}}$$

At this point we can check that $f(x) \geq 0$ for all x-values. ✓

So $k = \frac{5}{32}$ makes $f(x)$ a valid probability density function.

(3) Need to check two things:

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Check: $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{\infty} 0\,dx$

$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{1}{2}x\right]_1^2 + 0$

$= \frac{1}{2} + \left(1 - \frac{1}{2}\right)$

$= 1$ ✓

ii) $f(x)$ must be ≥ 0 for all x-values.

Check by looking at the definition of $f(x)$. ✓

(4) a) $P(0.5 \leq X \leq 2) = \int_{0.5}^{2} f(x)\,dx$

$= \int_{0.5}^{1} x\,dx + \int_{1}^{2} \frac{1}{2}\,dx$

$= \left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[\frac{1}{2}x\right]_1^2$

$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(1 - \frac{1}{2}\right)$

$= \frac{7}{8}$

(4) b) $P(X \le 1.5) = \int_{-\infty}^{1.5} f(x)\,dx$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{1.5} \tfrac{1}{2}\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{1}{2}x\right]_1^{1.5}$$

$$= \frac{1}{2} + \left(\frac{3}{4} - \frac{1}{2}\right)$$

$$= \frac{3}{4}$$

c) $P(X > 1.5) = 1 - P(X \le 1.5)$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

Alternatively : $P(X > 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

(5) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x^2\,dx + \int_{1}^{2} \frac{x}{2}\,dx + \int_{2}^{\infty} 0\,dx$

$= 0 + \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{4}\right]_1^2 + 0$

$= \frac{1}{3} + \left(1 - \frac{1}{4}\right)$

$= \frac{13}{12}$

(6) First find $E(X^2)$

then $\sigma^2 = E(X^2) - [E(X)]^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x^3\,dx + \int_{1}^{2} \frac{x^2}{2}\,dx + \int_{2}^{\infty} 0\,dx$

$= 0 + \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^3}{6}\right]_1^2 + 0$

$= \frac{1}{4} + \left(\frac{8}{6} - \frac{1}{6}\right)$

$= \frac{17}{12}$

$\sigma^2 = E(X^2) - [E(X)]^2$

$= \frac{17}{12} - \left(\frac{13}{12}\right)^2$

≈ 0.2431

(7) Need to check two things :

i) $\int_{-\infty}^{\infty} f(x)dx = 1$

Check : $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{3} 0dx + \int_{3}^{4}(x-3)dx + \int_{4}^{5}(5-x)dx + \int_{5}^{\infty} 0dx$

$= 0 + \left[\frac{x^2}{2} - 3x\right]_3^4 + \left[5x - \frac{x^2}{2}\right]_4^5 + 0$

$= 0 + \left(-4 - \left(\frac{-9}{2}\right)\right) + \left(\frac{25}{2} - 12\right) + 0$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$ ✓

ii) $f(x)$ must be $\geqslant 0$ for all values of x. Check by looking at the definition of $f(x)$. ✓

8) a) $P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} f(x)dx$

$$= \int_{3.5}^{4} (x-3)dx + \int_{4}^{4.5} (5-x)dx$$

$$= \left[\frac{x^2}{2} - 3x\right]_{3.5}^{4} + \left[5x - \frac{x^2}{2}\right]_{4}^{4.5}$$

$$= \left(-4 - \left(-\frac{35}{8}\right)\right) + \left(\frac{99}{8} - 12\right)$$

$$= \frac{3}{8} + \frac{3}{8}$$

$$= \frac{3}{4}$$

b) $P(X \le 4.5) = \int_{-\infty}^{4.5} f(x)dx$

$$= \int_{-\infty}^{3} 0\,dx + \int_{3}^{4} (x-3)dx + \int_{4}^{4.5} (5-x)dx$$

$$= 0 + \left[\frac{x^2}{2} - 3x\right]_{3}^{4} + \left[5x - \frac{x^2}{2}\right]_{4}^{4.5}$$

$$= \left(-4 - \left(-\frac{9}{2}\right)\right) + \left(\frac{99}{8} - 12\right)$$

$$= \frac{1}{2} + \frac{3}{8}$$

$$= \frac{7}{8}$$

c) $P(X > 4.5) = 1 - P(X \le 4.5)$

$$= 1 - \frac{7}{8}$$

$$= \frac{1}{8}$$

(9) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{3} 0\,dx + \int_{3}^{4} (x^2 - 3x)\,dx + \int_{4}^{5} (5x - x^2)\,dx + \int_{5}^{\infty} 0\,dx$$

$$= 0 + \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^4 + \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_4^5 + 0$$

$$= \left(\frac{-8}{3} - \left(\frac{-9}{2}\right)\right) + \left(\frac{125}{6} - \frac{56}{3}\right)$$

$$= \frac{11}{6} + \frac{13}{6}$$

$$= 4$$

(10) First find $E(X^2)$

Then $\sigma^2 = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{3} 0\,dx + \int_{3}^{4} [x^3 - 3x^2]\,dx + \int_{4}^{5} (5x^2 - x^3)\,dx + \int_{5}^{\infty} 0\,dx$$

$$= 0 + \left[\frac{x^4}{4} - x^3\right]_3^4 + \left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_4^5 + 0$$

$$= \left(0 - \left(-\frac{27}{4}\right)\right) + \left(\frac{625}{12} - \frac{128}{3}\right)$$

$$= \frac{27}{4} + \frac{113}{12}$$

$$= \frac{194}{12}$$

$$= \frac{97}{6}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \frac{97}{6} - (4)^2$$

$$= \frac{1}{6}$$

$$\sigma = \sqrt{\frac{1}{6}}$$

$$= \frac{\sqrt{6}}{6} \text{ or } \approx 0.408$$