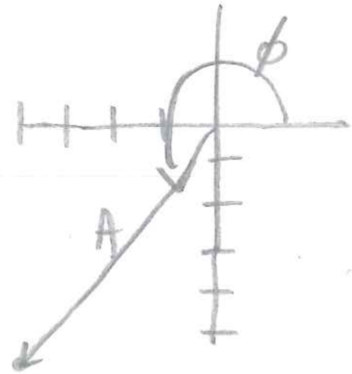
1. [4 marks] a) Write x in the form $A\sin(\frac{t}{\sqrt{5}}+\phi)$. Round A and ϕ to one decimal place.

$$x = -4\sin\tfrac{t}{\sqrt{5}} - 3\sqrt{3}\cos\tfrac{t}{\sqrt{5}}$$

$A\cos\phi = -4$
$A\sin\phi = -3\sqrt{3} \approx -5.2$



$A = \sqrt{(-4)^2 + (-3\sqrt{3})^2}$
≈ 6.6

$\phi = \tan^{-1}\left(\frac{-3\sqrt{3}}{-4}\right) + \pi$
≈ 4.1 rads

$$\boxed{x = 6.6\sin\left(\tfrac{t}{\sqrt{5}} + 4.1\right)}$$

2

b) Suppose the function x in part a) represents the position of a mass in a spring-mass system. When does the mass first pass through the equilibrium position?

$x = 0$

$\Rightarrow \sin\left(\frac{t}{\sqrt{5}} + 4.1\right) = 0$

$\Rightarrow \frac{t}{\sqrt{5}} + 4.1 = 0, \pi, 2\pi, \ldots$ 1

$\frac{t}{\sqrt{5}} + 4.1 = 0$ nonsense $(t<0)$

$\frac{t}{\sqrt{5}} + 4.1 = \pi$ "

$\frac{t}{\sqrt{5}} + 4.1 = 2\pi$ 1

$\boxed{t \approx 4.9 \text{ seconds}}$

2. [3 marks] Given the DE below, use $y = \sum_{n=0}^{\infty} c_n x^n$ to find the recurrence relation for the coefficients.

$xy'' - 4y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$xy'' - 4y = 0$$

$$x \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} - \sum_{n=0}^{\infty} 4 c_n x^n = 0$$

$\boxed{\begin{matrix} k = n-1 \\ k \geq 1 \end{matrix}}$ $\boxed{\begin{matrix} k = n \\ k \geq 0 \end{matrix}}$

start at $k=1$

$$\sum_{k=1}^{\infty} \quad - \quad k=0 \text{ term} \quad - \sum_{k=1}^{\infty} \quad = 0$$

$\boxed{n = k+1}$ $\boxed{1^{st} \text{ term}}$ $n = k$

$$\sum_{k=1}^{\infty} (k+1)k\, c_{k+1} x^k - 4c_0 - \sum_{k=1}^{\infty} 4 c_k x^k = 0$$

$\rightarrow$

2

2

$$\underbrace{-4C_0}_{\Rightarrow 0} + \sum_{k=1}^{\infty} \Big[\underbrace{(k+1)kC_{k+1} - 4C_k}_{=0}\Big]x^k = 0$$

Recurrence Relation:

$$\boxed{\begin{gathered} C_0 = 0 \\ C_{k+1} = \frac{4C_k}{(k+1)k}, \; k \geq 1 \end{gathered}}$$

1

3. [6 marks] Use the Laplace transform to solve the following DE.

$$y'' + 4y = 7\sin 3t, \quad y(0) = 9, \quad y'(0) = 4$$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{21}{s^2+9}$$

$$s^2 Y(s) - 9s - 4 + 4Y(s) = \frac{21}{s^2+9}$$ 1

2) Solve for $Y(s)$

$$(s^2+4)Y(s) = 9s + 4 + \frac{21}{s^2+9}$$

$$Y(s) = \frac{9s+4}{s^2+4} + \frac{21}{(s^2+4)(s^2+9)}$$ $\star$ 1

Partial Fractions

$$\frac{21}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\boxed{21 = (As+B)(s^2+9) + (Cs+D)(s^2+4)}$$

Sub $s=2i$: $21 = (2iA+B)(5)$

$A = 0$

$B = \frac{21}{5}$

2

Sub $s=3i$: $21 = (3iC+D)(-5)$

$C = 0$

$D = \frac{-21}{5}$

From (A) $Y(s) = \frac{9s+4}{s^2+4} + \frac{21}{5} \cdot \frac{1}{s^2+4} - \frac{21}{5} \cdot \frac{1}{s^2+9}$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = 9\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$
$$+ \frac{21}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$y(t) = 9\cos 2t + 2\sin 2t + \frac{21}{10}\sin 2t - \frac{7}{5}\sin 3t$$

or $y(t) = 9\cos 2t + \frac{41}{10}\sin 2t - \frac{7}{5}\sin 3t$

4. [6 marks] Find:

a) $\mathscr{L}^{-1}\{ \frac{6s-2}{s^2+8s+20} \}$

$= \mathscr{L}^{-1}\left\{ \frac{6(s+4)-26}{(s+4)^2+2^2} \right\}$

$= 6\mathscr{L}^{-1}\left\{ \frac{s+4}{(s+4)^2+2^2} \right\} - 13\mathscr{L}^{-1}\left\{ \frac{2}{(s+4)^2+2^2} \right\}$

$= 6e^{-4t}\cos 2t - 13e^{-4t}\sin 2t$

1

$s^2+8s+20 = (s+4)^2 + ?$

$= (s+4)^2 + 4$

$= (s+4)^2 + 2^2$

$6s-2 = ?(s+4) + ?$

$= 6(s+4) + ?$

$= 6(s+4) - 26$

2

~~b) $\mathscr{L}^{-1}\{ \frac{e^{-2s}}{s^4} \}$~~

b) $\mathscr{L}\{ \int_0^t \theta^8 e^{7(t-\theta)} d\theta \}$ $\longrightarrow$

$= \mathscr{L}\{ f * g \}$

$= F(s)G(s)$

$= \frac{8!}{s^9(s-7)}$

2

$f(\theta) = \theta^8$

$f(t) = t^8$

$F(s) = \frac{8!}{s^9}$

$g(t-\theta) = e^{7(t-\theta)}$

$g(t) = e^{7t}$

$G(s) = \frac{1}{s-7}$