1. [3 marks] Solve $x^3y''' + 8x^2y'' = 0$

Cauchy-Euler DE

$$m(m-1)(m-2) + 8m(m-1) = 0$$

$$m(m-1)[m-2+8] = 0$$

$$m(m-1)(m+6) = 0$$

$$m = 0, 1, -6$$

$$y = C_1 + C_2 x + C_3 x^{-6}$$

2. [4 marks] A mass weighing 29.4 N stretches a spring by 28 cm. The environment offers a damping force equivalent to β times the velocity, ~~in other words, $F = \beta$ N/(m/s)~~. Find β so that the spring-mass system is critically damped.

$$m = \frac{29.4}{9.8} = 3 \text{ kg}$$

$$k = \frac{29.4}{0.28} = 105 \text{ N/m}$$

DE: $\quad 3x'' + \beta x' + 105x = 0 \qquad (\beta > 0)$

auxiliary: $\quad 3m^2 + \beta m + 105 = 0$

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 1260}}{6}$$

critically damped $\Rightarrow$ repeated roots

$$\Rightarrow \beta^2 - 1260 = 0$$

$$\Rightarrow \beta = \sqrt{1260}$$

3. Use sigma notation to find C_2, C_3, C_4, C_5 and C_6 in terms of C_0 or C_1.

3. [6 marks] ~~Find the first six nonzero terms of the solution using sigma notation. All coefficients must be in terms of C_0 or C_1.~~

$y'' + 3xy = 0$

$$y = \sum_{n=0}^{\infty} C_n x^n \qquad y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 3x \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} 3 C_n x^{n+1} = 0$$ 1

$$\boxed{\begin{matrix} k = n-2 \\ k \geq 0 \end{matrix}} \qquad \boxed{\begin{matrix} k = n+1 \\ k \geq 1 \end{matrix}}$$

$\boxed{1^{st} \text{ term}}$

$$2C_2 + \sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^k + \sum_{k=1}^{\infty} 3 C_{k-1} x^k = 0$$

$\boxed{n = k+2}$ $\qquad \boxed{n = k-1}$

$$2C_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) C_{k+2} + 3 C_{k-1} \right] x^k = 0$$ 2

$$2C_2 = 0 \qquad (k+2)(k+1) C_{k+2} + 3 C_{k-1} = 0$$

$$\boxed{\begin{matrix} C_{k+2} = \dfrac{-3 C_{k-1}}{(k+2)(k+1)} \quad k \geq 1 \\ C_2 = 0 \end{matrix}}$$ 1

$$C_3 = -\frac{3C_0}{3 \cdot 2} = -\frac{C_0}{2} \qquad C_4 = \frac{-3C_1}{4 \cdot 3} = -\frac{C_1}{4}$$

$$C_5 = \frac{-3C_2}{5 \cdot 4} = 0 \qquad C_6 = \frac{-3C_3}{6 \cdot 5} = \frac{-C_3}{10} = \frac{C_0}{20}$$

4. [6 marks] The DE $y'' - 2y' + 2y = e^x \csc x$
has $y_c = C_1 e^x \cos x + C_2 e^x \sin x$.
Solve the DE using variation of parameters.

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \cos x \sin x + e^{2x} \sin^2 x$$

$$= e^{2x}$$ 1

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ e^x \csc x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= -e^{2x}$$ 1

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x \cos x - e^x \sin x & e^x \csc x \end{vmatrix}$$ 1

$$= e^{2x} \cot x$$

$$u_1' = \frac{W_1}{W} = -1 \Rightarrow u_1 = -x$$ 1

$$u_2' = \frac{W_2}{W} = \cot x \Rightarrow u_2 = \ln|\sin x|$$

1

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -x e^x \cos x + e^x \sin x \ln|\sin x|$$ 1

$$y = C_1 e^x \cos x + C_2 e^x \sin x - x e^x \cos x + e^x \sin x \ln|\sin x|$$