1. [4 marks] Find a second solution, given that $y_1 = e^{6x}$ is a solution:
$y'' - (\frac{1}{x} + 6)y' + \frac{6}{x}y = 0$

$$P(x) = -\left(\frac{1}{x} + 6\right)$$

$$e^{-\int P(x)dx} = e^{\int \left(\frac{1}{x}+6\right)dx}$$

$$= e^{\ln x + 6x}$$

$$= e^{\ln x} \cdot e^{6x}$$

$$= x e^{6x}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= e^{6x} \int \frac{x e^{6x}}{(e^{6x})^2} dx$$

$$= e^{6x} \int \frac{x e^{6x}}{e^{12x}} dx$$

$$= e^{6x} \int x e^{-6x} dx$$

$$= e^{6x}\left(\frac{-x e^{-6x}}{6} - \frac{e^{-6x}}{36}\right)$$

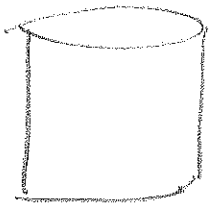
$$= -\frac{x}{6} - \frac{1}{36}$$

or $y_2 = 6x + 1$ etc.

	D	I
+	x	e^{-6x}
−	1	$-e^{-6x}/6$
		$e^{-6x}/36$

2. [4 marks] A 60L tank initially contains 125g of salt dissolved in 15L of water. An 8g/L salt solution is pumped in at a rate of 7L/min. The well-mixed solution is pumped out at a rate of 5L/min.

Let A represent the number of grams of salt in the tank after t minutes. Set up a DE involving A and t. Include any initial conditions but **DO NOT SOLVE THE DE.**



$\frac{8\text{g}}{\text{L}} \cdot \frac{7\text{L}}{\text{min}} = \frac{56\text{g}}{\text{min}}$

A = # grams after t mins

Volume $= 15 + 2t$

$\frac{A\text{ g}}{15+2t\text{ L}} \cdot \frac{5\text{L}}{\text{min}} = \frac{5A}{15+2t}$ g/min

$\frac{dA}{dt}$ = Inflow Rate − Outflow Rate (g/min)

$$\frac{dA}{dt} = 56 - \frac{5A}{15+2t}$$

$$A(0) = 125$$

3. [6 marks] Solve using Undetermined Coefficients (Section 4.4):

$y'' + 2y' - 24y = 7e^{4x} + 11x$

1) $y'' + 2y' - 24y = 0$

$m^2 + 2m - 24 = 0$

$(m+6)(m-4) = 0$

$m = -6, 4$

$y_c = C_1 e^{-6x} + C_2 e^{4x}$

2) $y_p = \cancel{Ae^{4x}} + Bx + C$ BAD CASE

$y_p = Axe^{4x} + Bx + C$

$y_p' = Ae^{4x} + 4Axe^{4x} + B$

$y_p'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x}$

$y_p'' = 16Axe^{4x} + 8Ae^{4x}$

3) $y_p \rightarrow DE \quad y'' + 2y' - 24y = 7e^{4x} + 11x$

$(16Axe^{4x} + 8Ae^{4x}) + 2(Ae^{4x} + 4Axe^{4x} + B)$

$-24(Axe^{4x} + Bx + C) = 7e^{4x} + 11x$

$0xe^{4x} + 10Ae^{4x} + 2B - 24Bx - 24C = 7e^{4x} + 11x$

→

$$10Ae^{4x} - 24Bx + (2B - 24C) = 7e^{4x} + 11x$$

$10A = 7$

$A = \frac{7}{10}$

$-24B = 11$

$B = \frac{-11}{24}$

$2B - 24C = 0$

$\frac{-22}{24} - 24C = 0$

$-24C = \frac{22}{24}$

$C = \frac{-22}{576}$

or $C = \frac{-11}{288}$

$y_p = Axe^{4x} + Bx + C$

$y_p = \frac{7}{10}xe^{4x} - \frac{11x}{24} - \frac{11}{288}$

4) $y = y_c + y_p$

$y = C_1e^{-6x} + C_2e^{4x} + \frac{7}{10}xe^{4x} - \frac{11x}{24} - \frac{11}{288}$

4. [6 marks] a) Solve $x^2y'' - 12xy' + 42y = 0$

Cauchy-Euler

$m(m-1) - 12m + 42 = 0$

$m^2 - m - 12m + 42 = 0$

$m^2 - 13m + 42 = 0$

$(m-6)(m-7) = 0$

$m = 6, 7$

$y_c = C_1 x^6 + C_2 x^7$

b) Solve $x^2y'' - 12xy' + 42y = 3x$ using Variation of Parameters (Section 4.6)

1) $y_1 = x^6 \quad y_2 = x^7$

2) $W = \begin{vmatrix} x^6 & x^7 \\ 6x^5 & 7x^6 \end{vmatrix} = x^{12}$

Standard form $y'' - \frac{12}{x}y' + \frac{42}{x^2}y = \frac{3}{x}$ ← $f(x)$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^7 \\ \frac{3}{x} & 7x^6 \end{vmatrix} = -3x^6$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^6 & 0 \\ 6x^5 & \frac{3}{x} \end{vmatrix} = 3x^5$

Extra Space for Question 4.

3) $u_1' = \frac{W_1}{W} = \frac{-3x^6}{x^{12}} = -3x^{-6}$

$u_1 = \int -3x^{-6}\,dx = \frac{3}{5}x^{-5}$

4) $u_2' = \frac{W_2}{W} = \frac{3x^5}{x^{12}} = 3x^{-7}$

$u_2 = \int 3x^{-7}\,dx = -\frac{1}{2}x^{-6}$

5) $y_p = u_1 y_1 + u_2 y_2$

$y_p = \frac{3}{5}x^{-5}(x^6) - \frac{1}{2}x^{-6}(x^7)$

$= \frac{3}{5}x - \frac{1}{2}x$

$= \frac{6}{10}x - \frac{5}{10}x$

$= \frac{x}{10}$

6) $y = y_c + y_p$

$y = C_1x^6 + C_2x^7 + \frac{x}{10}$