1. [5 marks] Find an implicit solution. Eliminate logarithms, absolute values and fractions from your solution.

$\frac{dy}{dx} = y^2 - 4$

$$\frac{dy}{y^2-4} = dx \qquad \text{Separable}$$

Partial Fractions

$$\frac{1}{y^2-4} = \frac{1}{(y+2)(y-2)} = \frac{A}{y+2} + \frac{B}{y-2}$$

$$1 = A(y-2) + B(y+2)$$

Sub $y=2$: $1 = 4B \Rightarrow B = \frac{1}{4}$

Sub $y=-2$: $1 = -4A \Rightarrow A = -\frac{1}{4}$

1

$$\left(-\frac{1}{4}\frac{1}{y+2} + \frac{1}{4}\frac{1}{y-2}\right)dy = dx$$

$$\int\left(-\frac{1}{4}\frac{1}{y+2} + \frac{1}{4}\frac{1}{y-2}\right)dy = \int dx$$

2

$$-\frac{1}{4}\ln|y+2| + \frac{1}{4}\ln|y-2| = x + C_1$$

$$\ln|y+2| - |y-2| = -4x + C_2$$

$$\ln\left|\frac{y+2}{y-2}\right| = -4x + C_2$$

$\rightarrow$

$$\left|\frac{y+2}{y-2}\right| = e^{-4x+C_2}$$

$$\frac{y+2}{y-2} = \pm e^{C_2} e^{-4x}$$

$$\frac{y+2}{y-2} = Ce^{-4x}$$

$$y+2 = Ce^{-4x}(y-2) \quad \text{or} \quad (y+2)e^{4x} = C(y-2)$$

2

2. [4 marks] Find an implicit solution:

$(4y\cos 4x + 3x^2e^y + \sin 6x)dx + (\sin 4x + x^3e^y + \frac{1}{1+y^2})dy = 0$

$M_y = 4\cos 4x + 3x^2e^y$

$N_x = 4\cos 4x + 3x^2e^y$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (4y\cos 4x + 3x^2e^y + \sin 6x)dx$

$= y\sin 4x + x^3e^y - \frac{1}{6}\cos 6x + g(y)$

AND

$f = \int (\sin 4x + x^3e^y + \frac{1}{1+y^2})dy$

$= y\sin 4x + x^3e^y + \arctan y + h(x)$

3

$\Rightarrow f = y\sin 4x + x^3e^y - \frac{1}{6}\cos 6x + \arctan y$

DE: $df = 0$

$\int df = \int 0$

Solution: $f = C$

$y\sin 4x + x^3e^y - \frac{1}{6}\cos 6x + \arctan y = C$

or $6y\sin 4x + 6x^3e^y - \cos 6x + 6\arctan y = C$

1

3. [6 marks] Find an explicit solution:

$\frac{2x}{3}\frac{dy}{dx} + y = \frac{e^{3x}}{\sqrt{x}}, \quad y(1) = 0$

Multiply by $\frac{3}{2x}$: $\quad \frac{dy}{dx} + \frac{3}{2x} y = \frac{3}{2x^{3/2}} e^{3x}$

DE is linear

$$P(x) = \frac{3}{2x} = \frac{3}{2}\frac{1}{x}$$

$$\begin{aligned} I.F. &= e^{\int \frac{3}{2}\frac{1}{x}\,dx} \\ &= e^{\frac{3}{2}\ln|x|} \\ &= e^{\ln|x|^{3/2}} \\ &= |x|^{3/2} \\ &= x^{3/2} \quad (x>0) \end{aligned}$$

1

1

$$x^{3/2}\frac{dy}{dx} + \frac{3}{2}x^{1/2} y = \frac{3}{2}e^{3x}$$

Integrate w.r.t. x:

2

$$x^{3/2} y = \frac{1}{2}e^{3x} + C$$

$y=0$, $x=1$:

$$0 = \frac{1}{2}e^{3} + C$$

$$C = -\frac{1}{2}e^{3}$$

$$x^{3/2} y = \frac{1}{2}e^{3x} - \frac{1}{2}e^{3}$$

2

$$y = \frac{\frac{1}{2}(e^{3x} - e^{3})}{x^{3/2}} \quad \text{or} \quad y = \frac{e^{3x} - e^{3}}{2x^{3/2}}$$

4. [4 marks] Solve the following homogeneous-degree DE using an appropriate substitution. Find an explicit solution.

$(x^2 + 2y^2)dx = 2xydy$

$$\begin{cases} y = ux \\ dy = udx + xdu \end{cases}$$ 1

$$(x^2 + 2u^2x^2)dx = 2xux(udx + xdu)$$

$$x^2dx + 2u^2x^2dx = 2u^2x^2dx + 2ux^3du$$

$$x^2dx = 2ux^3du$$

$$\frac{dx}{x} = 2udu$$ 1

$$\int \frac{dx}{x} = \int 2udu$$

$$\ln|x| = u^2 + C_1$$ 1

$$\ln|x| = \frac{y^2}{x^2} + C_1$$

$$x^2 \ln|x| = y^2 + C_1x^2$$

$$x^2 \ln|x| + Cx^2 = y^2$$

$$y = \pm\sqrt{x^2 \ln|x| + Cx^2}$$

or $y = \pm x\sqrt{\ln|x| + C}$ 1

5. [6 marks] Solve the following Bernoulli DE using an appropriate substitution. Find an implicit solution.

$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{4}e^x y^{-3}$

Bernoulli with $n = -3$

$$\begin{cases} y = u^{\frac{1}{1-n}} = u^{1/4} \\ \frac{dy}{dx} = \frac{1}{4}u^{-3/4}\frac{du}{dx} \end{cases}$$

$$\frac{1}{4}u^{-3/4}\frac{du}{dx} + \frac{1}{x}u^{1/4} = \frac{1}{4}e^x u^{-3/4}$$ 1

Mult. by $4u^{3/4}$:

$$\frac{du}{dx} + \frac{4}{x}u = e^x$$ 1

This is now linear

$P(x) = \frac{4}{x}$

$$\text{I.F.} = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = e^{\ln|x|^4} = |x|^4 = x^4$$ 1

$$x^4\frac{du}{dx} + 4x^3u = x^4e^x$$ 1

Integrate w.r.t. x →

Integration by Parts

	D	I
$+$	x^4	e^x
$-$	$4x^3$	
$+$	$12x^2$	
$-$	$24x$	
$+$	24	

$$x^4 u = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x + C$$

$y = u^{1/4}$

$y^4 = u$

$$x^4 y^4 = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x + C$$

2