1

$$(y + y\cos xy)dx + (x + x\cos xy + \tfrac{2}{y})dy = 0$$

$$M_y = 1 + \cos xy - xy\sin xy$$

$$N_x = 1 + \cos xy - xy\sin xy$$

$$M_y = N_x \Rightarrow \text{DE is exact}$$

$$f = \int (y + y\cos xy)\,dx$$

$$= xy + \sin xy + g(y)$$

AND

$$f = \int \left(x + x\cos xy + \tfrac{2}{y}\right)dy$$

$$= xy + \sin xy + 2\ln|y| + h(x)$$

$$\Rightarrow f = xy + \sin xy + 2\ln|y|$$

DE : $df = 0$

Solution : $f = C$

$$xy + \sin xy + 2\ln|y| = C$$ ✓

(2)

$$y' + \frac{y}{x} = x^3 y^3$$

Bernoulli $n=3$

Sub $\begin{cases} y = u^{\frac{1}{1-n}} = u^{-1/2} \\ \frac{dy}{dx} = -\frac{1}{2} u^{-3/2} \frac{du}{dx} \end{cases}$

$$-\frac{1}{2} u^{-3/2} \frac{du}{dx} + \frac{1}{x} u^{-1/2} = x^3 u^{-3/2}$$

Mult. by $-2u^{3/2}$:

$$\frac{du}{dx} - \frac{2}{x} u = -2x^3$$

Linear $\quad$ I.F. $= e^{\int -\frac{2}{x} dx}$

$= e^{-2\ln|x|}$

$= x^{-2} \qquad (x>0)$

$$x^{-2} \frac{du}{dx} - 2x^{-3} u = -2x$$

$$x^{-2} u = -x^2 + C$$

$$\boxed{\begin{array}{l} y = u^{-1/2} \\ u = y^{-2} \end{array}}$$

$$x^{-2} y^{-2} = -x^2 + C \checkmark$$

(3)

$$(x^2+y^2)\,dx - xy\,dy = 0$$

Hom. of degree 2 Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$$(x^2+u^2x^2)dx - x\,ux\,(u\,dx + x\,du) = 0$$

$$x^2\,dx - x^3u\,du = 0$$

$$x^2\,dx = x^3u\,du$$

$$\frac{dx}{x} = u\,du$$

$$\ln|x| = \frac{u^2}{2} + C_1$$

$\boxed{u=\frac{y}{x}}$ $$\ln|x| = \frac{1}{2}\frac{y^2}{x^2} + C_1$$

$$2x^2\ln|x| = y^2 + C_2x^2$$

$$y^2 = 2x^2\ln|x| + C_3x^2$$

$$y = \pm\sqrt{2x^2\ln|x| + Cx^2}$$

(4)

$$m \cdot a = -kv$$

$$m \frac{dv}{dt} = -kv$$

$$\frac{m\,dv}{v} = -k\,dt$$

$$\int \frac{m\,dv}{v} = \int -k\,dt$$

$$m \ln|v| = -kt + C_1 \qquad \ln|v| = -\frac{k}{m}t + C_2$$

$$|v| = e^{-\frac{k}{m}t + C_2}$$

$$v = \pm e^{C_2} e^{-\frac{k}{m}t}$$

$$v = C_3 e^{-\frac{k}{m}t}$$

$v = v_0$, $t = 0$: $\quad v_0 = C_3$

$$\boxed{v = v_0 e^{-\frac{k}{m}t}}$$

✓

⑤ Solve using Variation of Parameters

$$x^2y'' - xy' + y = x^3$$

1) y_c

Cauchy-Euler $\quad m(m-1) - m + 1 = 0$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_c = C_1 x + C_2 x \ln x$$

2) $W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x$

$$\boxed{\begin{array}{l} y'' - \dots = x \\ f(x) = x \end{array}}$$

$W_1 = \begin{vmatrix} 0 & x\ln x \\ x & \ln x + 1 \end{vmatrix} = -x^2 \ln x$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix} = x^2$

3) $u_1' = -x\ln x$

	D	I	
u	$\ln x$	$-x$	dv
du	$\frac{1}{x}$	$-\frac{x^2}{2}$	v

$u_1 = uv - \int v\,du$

$= -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx$

$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$

4) $u_2' = \frac{W_2}{W} = x$

$u_2 = \frac{x^2}{2}$

5) $y_p = u_1 y_1 + u_2 y_2$

$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)x + \frac{x^2}{2}(x\ln x)$

$= \frac{x^3}{4}$

6) $y = y_c + y_p$

$y = C_1 x + C_2 x\ln x + \frac{x^3}{4}$ ✓

(6) Find the general solution given $y_1 = x$

$x^3 y'' - x y' + y = 0$

$y'' - \frac{1}{x^2} y' + \frac{1}{x^3} y = 0$

$P(x) = -x^{-2}$

$-\int P(x)dx = \int x^{-2} dx$

$= -x^{-1}$

$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

$$= x\int \frac{e^{-x^{-1}}}{x^2}\,dx$$

$$\boxed{\begin{aligned} \text{Sub } u &= -x^{-1} \\ du &= x^{-2}\,dx \end{aligned}}$$

$$= x\int e^{u}\,du$$

$$= x\left[e^{u}\right]$$

$$= xe^{-x^{-1}}$$

$$y = C_1x + C_2xe^{-x^{-1}} \checkmark$$

7 Solve using Undetermined Coefficients

$$y''' - y'' = 2 + 2\cos x$$

1) $m^3 - m^2 = 0$

$m^2(m-1) = 0$

$m = 0, 0, 1$

$$y_c = C_1 + C_2x + C_3e^{x}$$

2) $y_p = A + B\cos x + C\sin x$

duplication

$$y_p = Ax^2 + B\cos x + C\sin x$$

3) $y_p = Ax^2 + B\cos x + C\sin x$

$y_p' = 2Ax - B\sin x + C\cos x$

$y_p'' = 2A - B\cos x - C\sin x$

$y_p''' = B\sin x - C\cos x$

$y''' - y'' = 2 + 2\cos x$

$B\sin x - C\cos x - 2A + B\cos x + C\sin x = 2 + 2\cos x$

$-2A = 2 \Rightarrow A = -1$

$B + C = 0$ (1)

$B - C = 2$ (2)

(1)+(2): $2B = 2$

$B = 1$

$C = -1$

$y_p = -x^2 + \cos x - \sin x$

4) $y = y_c + y_p$

$y = C_1 + C_2 x + C_3 e^x - x^2 + \cos x - \sin x$ ✓

⑧

a) Find $x(t)$

$$mx'' + \beta x' + kx = 0$$

$$x'' + 9x = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm 3i$$

$$\cancel{y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)}$$

$$x = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

$$\boxed{x = C_1 \cos 3t + C_2 \sin 3t}$$

$x = 2$, $t = 0$:

$$2 = C_1$$

$$\boxed{x = 2\cos 3t + C_2 \sin 3t}$$

$$x' = -6\sin 3t + 3C_2 \cos 3t$$

$x' = -1$, $t = 0$:

$$-1 = 3C_2$$

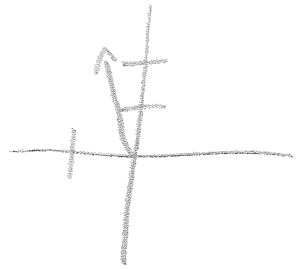
$$C_2 = -\frac{1}{3}$$

$$\boxed{x = 2\cos 3t - \frac{1}{3}\sin 3t}$$

b) Write as $A\sin(\omega t + \phi)$

$$x = 2\cos 3t - \frac{1}{3}\sin 3t$$

$A\sin\phi$ $\qquad$ $A\cos\phi$



$$A = \sqrt{2^2 + \left(\frac{-1}{3}\right)^2}$$

$$= \sqrt{4 + \frac{1}{9}}$$

$$= \sqrt{\frac{37}{9}}$$

$$= \frac{\sqrt{37}}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{\left(\frac{-1}{3}\right)}\right) \quad (+\pi\ ?)$$

$$= \tan^{-1}(-6) + \pi$$

$$\approx 1.74$$

$$x = \frac{\sqrt{37}}{3}\sin(3t + 1.74)$$

(9) Find first 2 nonzero terms of y_1 and y_2 using $y = \sum_{n=0}^{\infty} c_n x^n$

$$(x+2)y'' + y = 0$$

Solution :

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

DE : $$(x+2)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n-1$ | $k = n-2$ | $k = n$

$n = k+1$ | $n = k+2$ | $n = 0 \Rightarrow k = 0$

$n = 2 \Rightarrow k = 1$ | $n = 2 \Rightarrow k = 0$

Start at $k=1$.

$$\sum_{k=1}^{\infty} \overset{n=k+1}{(k+1)k c_{k+1}} x^k + 4c_2 x^0 + \sum_{k=1}^{\infty} \overset{n=k+2}{2(k+2)(k+1) c_{k+2}} x^k$$

$$+ c_0 x^0 + \sum_{k=1}^{\infty} \overset{n=k}{c_k} x^k = 0$$

$$(4C_2 + C_0) + \sum_{k=1}^{\infty} \left[(k+1)k\, C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k\right] x^k = 0$$

Set all coefficients $= 0$

$$4C_2 + C_0 = 0 \qquad (k+1)k\,C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k = 0$$

$$\boxed{C_2 = \frac{-C_0}{4}}$$

$$\boxed{\begin{gathered} C_{k+2} = \frac{-C_k - (k+1)k\,C_{k+1}}{2(k+2)(k+1)} \\ \text{Valid for } k \geq 1 \end{gathered}}$$

C_0: unknown

C_1: unknown

$C_2 = \frac{-C_0}{4}$

$$C_3 = \frac{-C_1 - 2C_2}{12} = \frac{-C_1}{12} - \frac{C_2}{6} = \frac{-C_1}{12} + \frac{C_0}{24}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$y = C_0 + C_1 x - \frac{C_0}{4} x^2 + \left(\frac{C_0}{24} - \frac{C_1}{12}\right) x^3 + \dots$$

$$y = C_0 \Big[\underbrace{1 - \frac{x^2}{4} + \frac{x^3}{24} + \dots}_{y_1}\Big] + C_1 \Big[\underbrace{x - \frac{x^3}{12} + \dots}_{y_2}\Big]$$

⑩ Solve using $\mathcal{L}$:

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1$$

where $f(t) = \begin{cases} \cos 2t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Solution: $f(t) = \square + \square\, \mathcal{U}(t-\pi)$

$$= \cos 2t - \cos 2t\, \mathcal{U}(t-\pi)$$

DE: $y'' + 4y = \cos 2t - \cos 2t\, \mathcal{U}(t-\pi)$

1) Apply $\mathcal{L}$:

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s} \underbrace{\mathcal{L}\{\cos 2(t+\pi)\}}$$

$= \mathcal{L}\{\cos(2t + 2\pi)\}$
$= \mathcal{L}\{\cos 2t \cos 2\pi - \sin 2t \sin 2\pi\}$
$= \mathcal{L}\{\cos 2t\}$

$$s^2 Y(s) - 1 + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s} \cdot \frac{s}{s^2+4}$$

2) Solve for $Y(s)$:

$$(s^2+4) Y(s) = 1 + \frac{s}{s^2+4} - \frac{s}{s^2+4} e^{-\pi s}$$

$$Y(s) = \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2} - \frac{s}{(s^2+4)^2} e^{-\pi s}$$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{4}\frac{4s}{(s^2+4)^2} - \frac{1}{4}\frac{4s}{(s^2+4)^2}e^{-\pi s} \right\}$$

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - f(t-\pi)\mathcal{U}(t-\pi)$$

$$\begin{cases} F(s) = \frac{1}{4}\frac{4s}{(s^2+4)^2} \\ f(t) = \frac{1}{4}t\sin 2t \\ f(t-\pi) = \frac{1}{4}(t-\pi)\sin 2(t-\pi) \end{cases}$$

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - \frac{1}{4}(t-\pi)[\sin 2(t-\pi)]\mathcal{U}(t-\pi)$$

Alternatively: $\sin 2(t-\pi)$ simplifies to $\sin 2t$

so $$y(t) = \frac{\sin 2t}{2} + \frac{t\sin 2t}{4} - \frac{(t-\pi)(\sin 2t)}{4}\mathcal{U}(t-\pi)$$

(11) $y'' + 4y = 3e^{-t} + \cos 2t, \ y(0)=1, \ y'(0)=2$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{3}{s+1} + \frac{s}{s^2+4}$$

$$s^2Y(s) - s - 2 + 4Y(s) = \frac{3}{s+1} + \frac{s}{s^2+4}$$

2) Solve for $Y(s)$

$$(s^2+4)Y(s) = s+2 + \frac{3}{s+1} + \frac{s}{s^2+4}$$

$$Y(s) = \frac{s+2}{s^2+4} + \frac{3}{(s+1)(s^2+4)} + \frac{s}{(s^2+4)^2}$$

3) Apply $\mathcal{L}^{-1}$

$$\star \quad y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2} + \frac{3}{(s+1)(s^2+4)}\right\}$$

4) Partial Fractions

$$\frac{3}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$3 = A(s^2+4) + (Bs+C)(s+1)$$

Sub $s=-1$: $3 = 5A \Rightarrow A = \frac{3}{5}$

s^2 coefficient: $0 = A+B \Rightarrow B = \frac{-3}{5}$

Constant: $3 = 4A+C \Rightarrow 3 = \frac{12}{5} + C \Rightarrow C = \frac{3}{5}$

5) Find $y(t)$

From ☆

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2} + \frac{3}{5}\cdot\frac{1}{s+1} - \frac{3}{5}\cdot\frac{s}{s^2+4} + \underbrace{\frac{3}{5}\cdot\frac{1}{s^2+4}}_{\frac{3}{10}\cdot\frac{2}{s^2+4}}\right\}$$

$$y(t) = \cos 2t + \sin 2t + \frac{t\sin 2t}{4} + \frac{3}{5}e^{-t} - \frac{3}{5}\cos 2t + \frac{3}{10}\sin 2t$$

OR $$y(t) = \frac{2}{5}\cos 2t + \frac{13}{10}\sin 2t + \frac{t\sin 2t}{4} + \frac{3}{5}e^{-t}$$

(12) $y'' + 4y' + 13y = 0, \quad y(0) = 7, \; y'(0) = 3$

1) Apply $\mathcal{L}$

$$s^2 Y(s) - s y(0) - y'(0) + 4[sY(s) - y(0)] + 13Y(s) = 0$$

$y(0) = 7$
$y'(0) = 3$: $\quad s^2 Y(s) - 7s - 3 + 4sY(s) - 28 + 13Y(s) = 0$

2) Solve for $Y(s)$: $\quad (s^2 + 4s + 13)\, Y(s) = 7s + 31$

$$Y(s) = \frac{7s + 31}{s^2 + 4s + 13}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{7s + 31}{s^2 + 4s + 13} \right\} \quad (*)$$

4) Complete the Square

$$\begin{aligned} s^2 + 4s + 13 &= (s+2)^2 + \, ? \\ &= (s+2)^2 + 9 \\ &= (s+2)^2 + 3^2 \end{aligned}$$

$$\begin{aligned} 7s + 31 &= \, ?\,(s+2) + \, ? \\ &= 7(s+2) + \, ? \\ &= 7(s+2) + 17 \end{aligned}$$

5) Find $y(t)$

From $\star$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{7(s+2)}{(s+2)^2+3^2} + \frac{17}{(s^2+2)^2+3^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ 7 \cdot \frac{s+2}{(s+2)^2+3^2} + \frac{17}{3} \cdot \frac{3}{(s+2)^2+3^2} \right\}$$

$$= 7e^{-2t}\cos 3t + \frac{17}{3}e^{-2t}\sin 3t$$

(13) Solve $\vec{X}' = \begin{bmatrix} 2 & 4 \\ 6 & 4 \end{bmatrix} \vec{X}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 6 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 24 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = -2, 8$$

$\lambda = -2$:

$$[A - \lambda I \mid \vec{0}]$$

$$[A + 2I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 6 & 6 & 0 \end{array}\right]$$

$R_1/4$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 6 & 6 & 0 \end{array}\right]$

$R_2 - 6R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\boxed{x_2 = t}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\boxed{x_1 = -t}$$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ Choose $\vec{K} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 8$:

$$[A - 8I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} -6 & 4 & 0 \\ 6 & -4 & 0 \end{array}\right]$$

$R_1/(-6)$ $\left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 6 & -4 & 0 \end{array}\right]$

$R_2 - 6R_1$ $\left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\boxed{x_2 = t}$$

$$x_1 - \frac{2}{3}x_2 = 0$$

$$x_1 = \frac{2}{3}x_2$$

$$\boxed{x_1 = \frac{2}{3}t}$$

$\vec{x} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} t$ Choose $\vec{K} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(or $\vec{K} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ etc.)

$$\vec{X} = C_1 \vec{K}_1 e^{\lambda_1 t} + C_2 \vec{K}_2 e^{\lambda_2 t}$$

$$\boxed{\vec{x} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{8t}}$$

(14) Solve $\vec{X}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{X}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 = 0$$

$$\lambda = 2, 2$$

$\lambda = 2$:

$$[A - 2I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_1 = t$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

Choose any nonzero $\vec{x}$: $\vec{K}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now we find $\vec{P}$ where $[A - \lambda I]\vec{P} = \vec{K}_1$.

$$[A - \lambda I \mid \vec{K}_1]$$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$p_1 = t$

$p_2 = 1$

$$\vec{p} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Choose any nonzero $\vec{p}$: $\vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now $\vec{X}_1 = \vec{K}_1 e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$

$$\vec{X}_2 = (\vec{K}_1 t + \vec{p}) e^{\lambda t} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

$$\vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2$$

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

(15) $\vec{X}' = \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix} \vec{X}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & -5 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 + 25 = 0$$

$$\lambda^2 + 2\lambda + 26 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{-100}}{2}$$

$$\lambda = \frac{-2 \pm 10i}{2}$$

$$\lambda = -1 \pm 5i \qquad (\alpha = -1,\ \beta = 5)$$

$\lambda = -1 + 5i$: $[A - \lambda I \mid \vec{0}]$

$$[A - (-1+5i) I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} -5i & -5 & 0 \\ 5 & -5i & 0 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 5 & -5i & 0 \\ -5i & -5 & 0 \end{array}\right]$$

$$\frac{R_1}{5}\left[\begin{array}{cc|c} 1 & -i & 0 \\ -5i & -5 & 0 \end{array}\right]$$

$$R_2+5iR_1 \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_2 = t$$

$$x_1 = it$$

$$\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix} t$$

Choose any nonzero $\vec{x}$: $\quad \vec{K}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\vec{K}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$$\vec{B}_1 = Re(\vec{K}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{B}_2 = Im(\vec{K}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{X}_1 = e^{\alpha t}[\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t] = e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos 5t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin 5t\right)$$

$$\vec{X}_2 = e^{\alpha t}[\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t] = e^{-t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 5t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos 5t\right)$$

$$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2$$

$$\vec{X} = e^{-t}\left(C_1\begin{bmatrix} -\sin 5t \\ \cos 5t \end{bmatrix} + C_2\begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}\right)$$