

① Find an implicit solution. Don't simplify solution.

$$(y + y \cos xy) dx + (x + x \cos xy + \frac{2}{y}) dy = 0$$

$$M_y = 1 + \cos xy - xy \sin xy$$

$$N_x = 1 + \cos xy - xy \sin xy$$

$$M_y = N_x \Rightarrow \text{DE is exact}$$

$$f = \int (y + y \cos xy) dx$$
$$= xy + \sin xy + g(y)$$

AN=

$$f = \int (x + x \cos xy + \frac{2}{y}) dy$$
$$= xy + \sin xy + 2 \ln|y| + h(x)$$

$$\Rightarrow f = xy + \sin xy + 2 \ln|y|$$

$$\text{DE: } df = 0$$

$$\text{Solution: } f = C$$

$$xy + \sin xy + 2 \ln|y| = C \checkmark$$

(2) Find an implicit solution. Don't simplify solution.

$$y' + \frac{y}{x} = x^3 y^3$$

Bernoulli $n=3$

$$\text{Sub. } y = u^{\frac{1}{1-n}} = u^{-1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2} u^{-3/2} \frac{du}{dx}$$

$$-\frac{1}{2} u^{-3/2} \frac{du}{dx} + \frac{1}{x} u^{-1/2} = x^3 u^{-3/2}$$

Mult. by $-2u^{3/2}$:

$$\frac{du}{dx} - \frac{2}{x} u = -2x^3$$

Linear

I.F.

$$= e^{\int -\frac{2}{x} dx}$$

$$= e^{-2 \ln|x|}$$

$$= x^{-2}$$

($x > 0$)

$$x^{-2} \frac{du}{dx} - 2x^{-3} u = -2x$$

$$x^{-2} u = -x^2 + C$$

$$\boxed{\begin{array}{l} y = u^{-1/2} \\ u = y^{-2} \end{array}}$$

$$x^{-2} y^{-2} = -x^2 + C \checkmark$$

③ Find an explicit solution

$$(x^2 + y^2) dx - xy dy = 0$$

Hom. of degree 2

Subst $y = ux$

$$dy = u dx + x du$$

$$(x^2 + u^2 x^2) dx - x u x (u dx + x du) = 0$$

$$x^2 dx - x^3 u du = 0$$

$$x^2 dx = x^3 u du$$

$$\frac{dx}{x} = u du$$

$$\ln|x| = \frac{u^2}{2} + C_1$$

$$\ln|x| = \frac{1}{2} \frac{y^2}{x^2} + C_1$$

$$2x^2 \ln|x| = y^2 + C_2 x^2$$

$$y^2 = 2x^2 \ln|x| + C_3 x^2$$

$$y = \pm \sqrt{2x^2 \ln|x| + Cx^2}$$

$u = \frac{y}{x}$

④ A speeding car has mass m and mass · acceleration is proportional to negative velocity. Find velocity $v(t)$ if the initial velocity is v_0 .

$$m \cdot a = -k v$$

$$m \frac{dv}{dt} = -k v$$

$$\frac{m dv}{v} = -k dt$$

$$\int \frac{m dv}{v} = \int -k dt$$

$$m \ln|v| = -kt + C_1 \quad \ln|v| = -\frac{k}{m}t + C_2$$

$$|v| = e^{-\frac{k}{m}t + C_2}$$

$$v = \pm e^{C_2} e^{-\frac{k}{m}t}$$

$$v = C_3 e^{-\frac{k}{m}t}$$

$$v = v_0 : \\ t = 0$$

$$v_0 = C_3 \\ \boxed{v = v_0 e^{-\frac{k}{m}t}} \quad \checkmark$$

⑤ Solve using Variation of Parameters

$$x^2 y'' - xy' + y = x^3$$

1) y_c

Cauchy-Euler

$$m(m-1) - m + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$y_c = C_1 x + C_2 x \ln x$$

$$2) W = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x$$

$$\boxed{y'' - \dots = x}$$

$$W_1 = \begin{vmatrix} 0 & x \ln x \\ x & \ln x + 1 \end{vmatrix} = -x^2 \ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix} = x^2$$

$$3) u_1' = -x \ln x$$

$$\begin{aligned} u_1 &= W_1^{-1} \int W_1 f(x) dx \\ &= \frac{-x^2}{2} \ln x + \int \frac{x}{2} dx \\ &= \frac{-x^2}{2} \ln x + \frac{x^2}{4} \end{aligned}$$

| D | I |
|---------------|------------------|
| $\ln x$ | $-x$ |
| $\frac{1}{x}$ | $-\frac{x^2}{2}$ |

$$4) u_2' = \frac{W_2}{W} = x$$

$$u_2 = \frac{x^2}{2}$$

$$\begin{aligned} 5) y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-\frac{x^2}{2} \ln x + \frac{x^2}{4} \right) x + \frac{x^2}{2} (x \ln x) \\ &= \frac{x^3}{4} \end{aligned}$$

$$6) y = y_c + y_p$$

$$y = C_1 x + \left(2x \ln x + \frac{x^3}{4} \right) \checkmark$$

⑥ Find the general solution given $y_1 = x$

$$x^3 y'' - x y' + y = 0$$

$$y'' - \frac{1}{x^2} y' + \frac{1}{x^3} y = 0$$

$$P(x) = -x^{-2}$$

$$\begin{aligned} -\int P(x) dx &= \int x^{-2} dx \\ &= -x^{-1} \end{aligned}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x) dx}}{y_1^2} dx$$

$$= x \int \frac{e^{-x^{-1}}}{x^2} dx$$

$$= x \int e^u du$$

$$= x [e^u]$$

$$= x e^{-x^{-1}}$$

$$y = C_1 x + C_2 x e^{-x^{-1}}$$

⑦ Solve using Undetermined Coefficients

$$y''' - y'' = 2 + 2 \cos x$$

1)

$$m^3 - m^2 = 0$$

$$m^2(m-1) = 0$$

$$m = 0, 0, 1$$

$$y_c = C_1 + C_2 x + C_3 e^x$$

2)

$$y_p = A + B \cos x + C \sin x$$

← duplicate

$$y_p = Ax^2 + B \cos x + C \sin x$$

$$3) \quad y_p = Ax^2 + B \cos x + C \sin x$$

$$y_p' = 2Ax - B \sin x + C \cos x$$

$$y_p'' = 2A - B \cos x - C \sin x$$

$$y_p''' = B \sin x - C \cos x$$

$$y''' - y'' = 2 + 2 \cos x$$

$$\underline{B \sin x} - \underline{C \cos x} - \underline{2A} + \underline{B \cos x} + \underline{C \sin x} = 2 + 2 \cos x$$

$$-2A = 2 \quad \Rightarrow \quad A = -1$$

$$B + C = 0 \quad (1)$$

$$B - C = 2 \quad (2)$$

$$(1) + (2): \quad 2B = 2$$

$$B = 1$$

$$C = -1$$

$$y_p = -x^2 + \cos x - \sin x$$

$$4) \quad y = y_c + y_p$$

$$y = C_1 + C_2 x + C_3 e^x - x^2 + \cos x - \sin x \quad \checkmark$$

⑧ 1kg mass, no damping,
spring constant = 9 N/m.

Mass is initially 2m below eq. position,
moving upward at 1 m/s.

a) Find $x(t)$

$$m x'' + \beta x' + kx = 0$$

$$x'' + 9x = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm 3i$$

~~$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$~~

~~$$x = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t)$$~~

$$x = C_1 \cos 3t + C_2 \sin 3t$$

$$2 = C_1$$

$$x = 2 \cos 3t + C_2 \sin 3t$$

$$x' = -6 \sin 3t + 3C_2 \cos 3t$$

$$x' = -1$$

$$t = 0$$

$$-1 = 3C_2$$

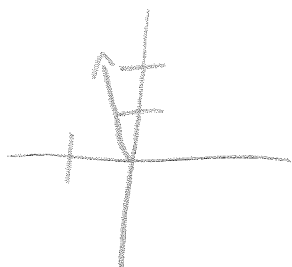
$$C_2 = -\frac{1}{3}$$

$$x = 2 \cos 3t - \frac{1}{3} \sin 3t$$

b) Write as $A \sin(\omega t + \phi)$

$$x = 2 \cos 3t - \frac{1}{3} \sin 3t$$

\uparrow \uparrow
 $A \sin \phi$ $A \cos \phi$



$$A = \sqrt{2^2 + \left(-\frac{1}{3}\right)^2}$$

$$= \sqrt{4 + \frac{1}{9}}$$

$$= \sqrt{\frac{37}{9}}$$

$$= \frac{\sqrt{37}}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{\left(-\frac{1}{3}\right)}\right) (+\pi?)$$

$$= \tan^{-1}(-6) + \pi$$

$$\approx 1.74$$

$$x = \frac{\sqrt{37}}{3} \sin(3t + 1.74) \quad \checkmark$$

⑨ Find first 2 nonzero terms of y_1 and y_2 using $y = \sum_{n=0}^{\infty} c_n x^n$

$$(x+2)y'' + y = 0$$

Solution:

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$\text{DE: } (x+2) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\boxed{\begin{matrix} k=n-1 \\ k \geq 1 \end{matrix}}$$

$$\boxed{\begin{matrix} k=n-2 \\ k \geq 0 \end{matrix}}$$

$$\boxed{\begin{matrix} k=n \\ k \geq 0 \end{matrix}}$$

Start at $k=1$

$$\sum_{k=1}^{\infty} \left(\begin{matrix} \text{1st term} \\ (k=0 \text{ term}) \end{matrix} + \sum_{k=1}^{\infty} \right) + \begin{matrix} \text{1st term} \\ (k=0 \text{ term}) \end{matrix} + \sum_{k=1}^{\infty} = 0$$

$$\sum_{k=1}^{\infty} (k+1)k c_{k+1} x^k + 4c_2 x^0 + \sum_{k=1}^{\infty} 2(k+2)(k+1) c_{k+2} x^k$$

$$+ c_0 x^0 + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$(4C_2 + C_0) + \sum_{k=1}^{\infty} [(k+1)k C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k] x^k = 0$$

Set all coefficients = 0

$$4C_2 + C_0 = 0$$

$$\boxed{C_2 = -\frac{C_0}{4}}$$

$$(k+1)k C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k = 0$$

$$\boxed{C_{k+2} = \frac{-C_k - (k+1)k C_{k+1}}{2(k+2)(k+1)}} \\ \text{valid for } k \geq 1$$

C_0 : unknown

C_1 : unknown

$$C_2 = -\frac{C_0}{4}$$

$$C_3 = \frac{-C_1 - 2C_2}{12} = -\frac{C_1}{12} - \frac{C_2}{6} = -\frac{C_1}{12} + \frac{C_0}{24}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$y = C_0 + C_1 x - \frac{C_0}{4} x^2 + \left(\frac{C_0}{24} - \frac{C_1}{12}\right) x^3 + \dots$$

$$y = C_0 \left[1 - \frac{x^2}{4} + \frac{x^3}{24} + \dots \right] + C_1 \left[x - \frac{x^3}{12} + \dots \right]$$

$\underbrace{\hspace{10em}}_{y_1} \qquad \qquad \qquad \underbrace{\hspace{10em}}_{y_2}$

(10) Solve using \mathcal{L} :

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1$$

$$\text{where } f(t) = \begin{cases} \cos 2t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

$$\text{Solution: } f(t) = \square + \square \mathcal{U}(t-\pi) \\ = \cos 2t - \cos 2t \mathcal{U}(t-\pi)$$

$$\text{DE: } y'' + 4y = \cos 2t - \cos 2t \mathcal{U}(t-\pi)$$

1) Apply \mathcal{L} :

$$\mathcal{L}^2 Y(s) - \mathcal{L}y(0) - y'(0) + 4Y(s) = \frac{1}{s^2+4} - e^{-\pi s} \underbrace{\mathcal{L}\{\cos 2(t+\pi)\}}_{\substack{= \mathcal{L}\{\cos(2t+2\pi)\} \\ = \mathcal{L}\{\cos 2t\} \\ \text{Since period} = 2\pi}}$$

$$\mathcal{L}^2 Y(s) - 1 + 4Y(s) = \frac{1}{s^2+4} - e^{-\pi s} \cdot \frac{1}{s^2+4}$$

2) Solve for $Y(s)$:

$$(s^2+4)Y(s) = 1 + \frac{1}{s^2+4} - \frac{1}{s^2+4} e^{-\pi s}$$

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{(s^2+4)^2} - \frac{1}{(s^2+4)^2} e^{-\pi s}$$

3) Apply \mathcal{L}^{-1} :

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{2} \frac{2}{s^2+4} + \frac{1}{4} \frac{4s}{(s^2+4)^2} - \frac{1}{4} \frac{4s}{(s^2+4)^2} e^{-\pi/2} \right\}$$

$$y(t) = \frac{1}{2} \sin 2t + \frac{1}{4} t \sin 2t - f(t-\pi) u(t-\pi)$$

$$\begin{cases} F(s) = \frac{1}{4} \frac{4s}{(s^2+4)^2} \\ f(t) = \frac{1}{4} t \sin 2t \\ f(t-\pi) = \frac{1}{4} (t-\pi) \sin 2(t-\pi) \end{cases}$$

$$y(t) = \frac{1}{2} \sin 2t + \frac{1}{4} t \sin 2t - \frac{1}{4} (t-\pi) \sin 2(t-\pi) u(t-\pi)$$

(11) Solve $\vec{X}' = \begin{bmatrix} 2 & 4 \\ 6 & 4 \end{bmatrix} \vec{X}$

Solution:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 6 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 24 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = -2, 8$$

$\lambda = -2$:

$$[A - \lambda I | \vec{0}]$$

$$[A + 2I | \vec{0}]$$

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 6 & 6 & 0 \end{array} \right]$$

$R_2/4$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 6 & 6 & 0 \end{array} \right]$$

$R_2 - 6R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\boxed{x_2 = t}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\boxed{x_1 = -t}$$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t$$

Choose $\vec{K} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\lambda = 8:$$

$$[A - 8I \mid \vec{b}]$$

$$\begin{bmatrix} -6 & 4 & | & 0 \\ 6 & -4 & | & 0 \end{bmatrix}$$

$$R_1 \leftarrow (-6) \quad \begin{bmatrix} 1 & -2/3 & | & 0 \\ 6 & -4 & | & 0 \end{bmatrix}$$

$$R_2 - 6R_1 \quad \begin{bmatrix} 1 & -2/3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\boxed{x_2 = t}$$

$$x_1 - \frac{2}{3}x_2 = 0$$

$$x_1 = \frac{2}{3}x_2$$

$$\boxed{x_1 = \frac{2}{3}t}$$

$$\vec{x} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} t$$

$$\text{Choose } \vec{K} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{(or } \vec{K} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \text{ etc.)}$$

$$\vec{x} = C_1 \vec{K}_1 e^{\lambda_1 t} + C_2 \vec{K}_2 e^{\lambda_2 t}$$

$$\boxed{\vec{x} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{8t}}$$