Math 252 Practice Problems

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1. Find the explicit solution of the initial value problem:
   $\frac{dP}{dt} = P - P^2$, $P(0) = 2$.

2. Find the solution of the initial value problem:
   $x^3 \frac{dy}{dx} + 4x^2 y = e^{-x}$, $y(-1) = 0$.

3. Find the solution of the initial value problem:
   $xy' + 2y = 3x$, $y(1) = 0$.

4. Solve $\frac{dy}{dx} = \frac{y^2}{xy+x^2}$.

5. Solve the differential equation $y' = \frac{x+3y}{3x+y}$.

6. Solve the differential equation: $y' = y(xy^3 - 1)$.
Give an explicit solution.

7. Find the value of the constant k so that the differential equation
   $(y^2 \cos x - 3x^2 y - 2x)dx + (ky \sin x - x^3 + 4y)dy = 0$
is exact and then solve the equation.

8. Solve $\frac{dy}{dx} = \frac{1-x-y}{x+y}$.

9. Solve the differential equation $x^2 y' = xy + y^4$.

10. Solve the following differential equation:
   $(1 + xye^x)dx + (xe^x + x \cos y)dy = 0$.

11. A tank contains 200 litres of fluid in which 100 grams of salt is dissolved initially. Brine containing 5 grams of salt per litre is then pumped into the tank at a rate of 4 litres per minute; the well-mixed solution is pumped out at the same rate.
a) Find the mass $m(t)$ of salt in the tank as a function of time t.
b) Find $\lim_{t\to\infty} m(t)$.
c) After how much time do we have 600 grams of salt in the tank?

12. According to Newton's Law of Cooling, the rate of change of temperature of a body is proportional to the difference in temperature between the body and the surrounding air, i.e. $\frac{dT}{dt} = -k(T - T_e)$, where T is the body temperature, T_e is the surrounding air temperature and $k > 0$ is a constant.

a) Solve the differential equation to find $T(t)$, assuming T_e is constant and $T(0) = T_0$.

b) In a murder investigation, a corpse was found by Sherlock Holmes at exactly 8:00pm. Being alert, he immediately measures the temperature of the body and finds it to be 25°C. Two hours later, Holmes again measures the temperature of the corpse and finds it to be 15°C. If the room temperature is 10°C, when did the murder occur? Assume that the temperature of the body at the time of the murder was 37°C.

13. Find the form of a particular solution y_p to the following. Leave your answer with undetermined coefficients.
a) $y'' + 4y = 5x + \sin(2x)$
b) $y'' - 2y' + y = 3x^2 + 5xe^x$
c) $y'' + 2y' + 5y = 3e^{-x} + 4e^{-x}\sin(2x) - 2\cos(3x)$

14. Find the solution of the initial value problem
$$y'' - 4xy' + (4x^2 - 2)y = 0,\ y(0) = 1,\ y'(0) = 2$$
given that $y_1 = e^{x^2}$ is a solution of the equation.

15. Use the method of undetermined coefficients to find the general solution of $y'' + y' - 6y = 4e^{3x} + 5\cos(2x)$.

16. Use the method of undetermined coefficients to find the general solution of $y'' - 5y' + 6y = x + 2 + 3e^{2x}$.

17. Solve $y'' + 4y = \sec(2x)$.

18. Consider the differential equation $x^2y'' - 3xy' + ky = 0$ $(x > 0)$.
Find the general solution if:
a) $k = -5$
b) $k = 4$
c) $k = 13$

19. Use the method of variation of parameters to find the general solution of
$$x^2y'' - 2xy' + 2y = x^4e^{-x}.$$

20. Find the solution of the initial value problem
$x^2y'' - xy' + y = 2x$, $y(1) = 2$, $y'(1) = 3$.

21. A mass of 2kg is attached to a spring whose spring constant is 3 N/m. The medium offers a damping force of magnitude 5 times the instantaneous velocity.

a) Determine the steady-state position of the mass if it is driven by an external force (in Newtons) of magnitude $F(t) = 4\cos(2t) + 3\sin(2t)$.
b) Express your answer to a) in the form $x(t) = A\sin(\omega t + \phi)$.

22. A 1kg mass is attached to a spring with spring constant 2 N/m and a damping device with $\beta = 3$ N/(m/s).

a) Find the position $x(t)$ of the mass if it is released 50 cm below equilibrium with no initial velocity.

b) A force (in Newton) of the form $F(t) = F_0 \cos t$ is applied to the system. Find the positive value of F_0 so that the amplitude of the steady-state solution equals 1 m.

23. A mass of 1kg is attached to a spring whose spring constant is 5 N/m. The medium offers a damping force of magnitude 4 times the instantaneous velocity.

a) Find the position $x(t)$ of the mass if it is released 50 cm below equilibrium with an initial downward velocity of 1 m/s.

b) Find all times $t > 0$ when the mass passes through the equilibrium position.

24. A mass of 1kg is attached to a spring whose spring constant is 6 N/m. The medium offers a damping force of magnitude 5 times the instantaneous velocity. An external force (in Newton) of the form $F_e = 3\sin(2t) - \cos(2t)$ is applied to the system. Find the position $x(t)$ of the mass if it is released 50 cm above equilibrium with a downward velocity of 1 m/s.

25. Find the first five nonzero terms of a power series solution of the initial value problem:
$y'' - x^2y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$.
For which values of x does the series converge?

26. Find the first five nonzero terms of a power series solution of the initial value problem:
$(x+1)y'' - (2-x)y' + y = 0$, $y(0) = 2$, $y'(0) = -1$.
For which values of x does the series converge?

27. Use $y = C_0 + C_1x + C_2x^2 + \ldots$ to solve the differential equation: $y'' + e^x y = 0$. Find the first three nonzero terms of y_1 and y_2.

28. Solve $2xy'' + 5y' + xy = 0$ using a series.

29. Evaluate the following:

a) $\mathscr{L}\{f(t)\}$ for $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 2, & 2 \le t < 4 \\ -1, & t \ge 4 \end{cases}$

b) $\mathscr{L}^{-1}\{\frac{e^{-2s}}{s+3}\}$

c) $\mathscr{L}\{te^{-2t}\sin(3t)\}$

d) $\mathscr{L}^{-1}\{\frac{s+4}{s^2+2s+5}\}$

e) $\mathscr{L}\{\cos t \cdot \mathscr{U}(t-\pi)\}$

30. Use the Laplace transform to solve the initial value problem
$y''(t) + 2y'(t) + 5y(t) = \cos t$, $y(0) = 2$, $y'(0) = -1$.

31. Use the Laplace transform to solve the initial value problem
$y''(t) + 4y(t) = 2e^{-t} + 3\sin(2t)$, $y(0) = 2$, $y'(0) = 1$. Simplify your answer.

32. Use the Laplace transform to solve the initial value problem
$y''(t) + 2y'(t) + 5y(t) = 3e^{-t}\cos(2t)$, $y(0) = 1$, $y'(0) = 2$.

33. Use the Laplace transform to solve the initial value problem
$y''(t) + y(t) = \cos t + \delta(t-\pi)$, $y(0) = 0$, $y'(0) = 1$.

34. Solve the initial value problem $y''(t) + 4y(t) = f(t)$, $y(0) = 0$, $y'(0) = 2$, where

$$f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 2, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

35. The differential equation for the current $i(t)$ in an LR series circuit is $L\frac{di}{dt} + Ri = V(t)$. Find the current $i(t)$ if $i(0) = 0$, the inductance is $L = 1$ H, the resistance is $R = 10\ \Omega$ and $V(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

36. Solve the initial value problem
$y''(t) + y(t) = 2e^{-t} + \delta(t - \frac{\pi}{2}) + \delta(t-\pi)$, $y(0) = 2$, $y'(0) = 1$.
Write your answer as a piecewise-defined function.

37. Evaluate $\mathscr{L}^{-1}\{\frac{1}{(s-3)(s+4)}\}$ by using a convolution.

38. Evaluate $\mathscr{L}^{-1}\{\frac{1}{(s^2+1)^2}\}$ by using a convolution.