(29) a) $f(t) = 1 + 1\cdot\mathcal{U}(t-2) - 3\cdot\mathcal{U}(t-4)$

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + e^{-2s}\mathcal{L}\{1\} + e^{-4s}\mathcal{L}\{-3\}$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s} \quad \frac{-3e^{-4s}}{s}$$

b) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} = f(t-2)\mathcal{U}(t-2)$ $\quad f(t) = e^{-3t}$

$$= e^{-3(t-2)}\mathcal{U}(t-2)$$

c) $\mathcal{L}\{te^{-2t}\sin(3t)\} = F(s+2)$ $\quad f(t) = t\sin(3t)$, $F(s) = \frac{6s}{(s^2+9)^2}$

$$= \frac{6(s+2)}{((s+2)^2+9)^2}$$

d) $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+4}\right\} \leftarrow$ complete the square

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4} + \frac{3}{2}\cdot\frac{2}{(s+1)^2+4}\right\}$$

$$= e^{-t}\cos(2t) + \frac{3}{2}\cdot e^{-t}\sin(2t)$$

e). $\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi)\} = e^{-\pi s}\, \mathcal{L}\{\cos(t+\pi)\}$

$$= e^{-\pi s}\, \mathcal{L}\{-\cos t\}$$

$$= -e^{-\pi s} \cdot \frac{s}{s^2+1}$$

$$= -\frac{s e^{-\pi s}}{s^2+1}$$

$\cos(t+\pi) = -\cos t$ either from the graph of $f(t) = \cos t$

or $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(30) $\mathcal{L}\{y''(t) + 2y'(t) + 5y(t)\} = \mathcal{L}\{\cos t\}$ $\quad y(0)=2$

$y'(0)=-1$

$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 5Y(s) = \frac{s}{s^2+1}$

$(s^2+2s+5)Y(s) - 2s + 1 - 4 = \frac{s}{s^2+1}$

$(s^2+2s+5)Y(s) = \frac{s}{s^2+1} + 2s + 3$

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+5)} + \frac{2s+3}{s^2+2s+5}$$

$$\frac{s}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$s = (As+B)(s^2+2s+5) + (Cs+D)(s^2+1)$

Sub $s=i$ $\quad i = (Ai+B)(2i+4)$

$i = (-2A+4B) + (2B+4A)i$

$\left.\begin{array}{l} -2A+4B=0 \\ 4A+2B=1 \end{array}\right\} \begin{array}{l} A=\frac{1}{5} \\ B=\frac{1}{10} \end{array}$

$[s^3]$: $\quad 0 = A + C$

$C = -1/5$

$[1]$: $\quad 0 = 5B + D$

$D = -1/2$

See next page →

$$Y(s) = \frac{\frac{1}{5}s + \frac{1}{10}}{s^2+1} + \frac{-\frac{1}{5}s - \frac{1}{2}}{s^2+2s+5} + \frac{2s+3}{s^2+2s+5}$$

$$Y(s) = \frac{\frac{1}{5}s + \frac{1}{10}}{s^2+1} + \frac{\frac{9}{5}s + \frac{5}{2}}{s^2+2s+5}$$

complete the square

$$Y(s) = \frac{1}{5} \cdot \frac{s}{s^2+1} + \frac{1}{10} \cdot \frac{1}{s^2+1} + \frac{\frac{9}{5}(s+1) + \frac{7}{10}}{(s+1)^2+4}$$

$$Y(s) = \frac{1}{5} \cdot \frac{s}{s^2+1} + \frac{1}{10} \cdot \frac{1}{s^2+1} + \frac{9}{5} \cdot \frac{s+1}{(s+1)^2+4} + \frac{7}{20} \cdot \frac{2}{(s+1)^2+4}$$

Applying $\mathcal{L}^{-1}$:

$$y(t) = \frac{1}{5}\cos t + \frac{1}{10}\sin t + \frac{9}{5}e^{-t}\cos(2t) + \frac{7}{20}e^{-t}\sin(2t)$$

(31) $\mathcal{L}\{y''(t)+4y(t)\} = \mathcal{L}\{2e^{-t}+3\sin(2t)\}$ $\qquad y(0)=2$, $y'(0)=1$

$$\underline{s^2Y(s)-sy(0)-y'(0)} \quad +4Y(s) = \frac{2}{s+1} + \frac{6}{s^2+4}$$

$$(s^2+4)Y(s) \quad -2s-1 = \frac{2}{s+1} + \frac{6}{s^2+4}$$

$$Y(s) = \frac{2}{(s+1)(s^2+4)} + \frac{6}{(s^2+4)^2} + \frac{2s+1}{s^2+4}$$

$$\frac{2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$2 = A(s^2+4) + (Bs+C)(s+1)$$

Sub $s=-1$: $\quad 2 = 5A \qquad A = 2/5$

$[s^2]$: $\quad 0 = A+B \qquad B = -2/5$

$[1]$: $\quad 2 = 4A+C$

$$\frac{10}{5} = \frac{8}{5}+C \qquad C = 2/5$$

$$Y(s) = \frac{2}{5}\cdot\frac{1}{s+1} + \frac{-2/5\,s + 2/5}{s^2+4} + \frac{6}{(s^2+4)^2} + \frac{2s+1}{s^2+4}$$

$$Y(s) = \frac{2}{5}\cdot\frac{1}{s+1} + \frac{8/5\,s + 7/5}{s^2+4} + \frac{6}{(s^2+4)^2}$$

$$Y(s) = \frac{2}{5}\cdot\frac{1}{s+1} + \frac{8}{5}\cdot\frac{s}{s^2+4} + \frac{7}{10}\cdot\frac{2}{s^2+4} + \frac{6}{(s^2+4)^2}$$

Applying $\mathcal{L}^{-1}$:

$$y(t) = \frac{2}{5}e^{-t} + \frac{8}{5}\cos(2t) + \frac{7}{10}\sin(2t) + 6\cdot\left(\frac{\sin 2t - 2t\cos 2t}{16}\right)$$

using $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2^2)^2}\right\}$

$$y(t) = \frac{2}{5}e^{-t} + \left(\frac{8}{5} - \frac{3}{4}t\right)\cos(2t) + \frac{43}{40}\sin(2t)$$

$\frac{7}{10} + \frac{3}{8} = \frac{86}{80} = \frac{43}{40}$

(32) $\mathcal{L}\{y''(t)+2y'(t)+5y(t)\}=\mathcal{L}\{3e^{-t}\cos(2t)\}$ $\quad y(0)=1$, $y'(0)=2$

$\underline{s^2Y(s)-sy(0)-y'(0)}+2(sY(s)-y(0))+5Y(s)=\dfrac{3(s+1)}{(s+1)^2+4}$

$F(s+1)$

$F(s)=\dfrac{s}{s^2+4}$

$(\underbrace{s^2+2s+5}_{(s+1)^2+4})Y(s)-s-2-2=\dfrac{3(s+1)}{(s+1)^2+4}$

$$Y(s)=\frac{3(s+1)}{((s+1)^2+4)^2}+\frac{s+4}{(s+1)^2+4}$$

$$Y(s)=\frac{3(s+1)}{((s+1)^2+4)^2}+\frac{(s+1)}{(s+1)^2+4}+\frac{3}{(s+1)^2+4}$$

$$Y(s)=\frac{3}{4}\cdot\frac{4(s+1)}{((s+1)^2+2^2)^2}+\frac{s+1}{(s+1)^2+2^2}+\frac{3}{2}\cdot\frac{2}{(s+1)^2+2^2}$$

Applying $\mathcal{L}^{-1}$:

$$y(t)=\frac{3}{4}te^{-t}\sin(2t)+e^{-t}\cos(2t)+\frac{3}{2}\cdot e^{-t}\sin(2t)$$

$e^{-t}f(t)$

$f(t)=t\sin(2t)$

$$y(t)=\frac{3}{4}te^{-t}\sin(2t)+e^{-t}\cos(2t)+\frac{3}{2}\cdot e^{-t}\sin(2t)$$

(33) $\mathcal{L}\{y''(t)+y(t)\} = \mathcal{L}\{\cos t + \delta(t-\pi)\}$ $\quad y(0)=0$, $y'(0)=1$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2+1} + e^{-\pi s}$$

$$(s^2+1)Y(s) - 1 = \frac{s}{s^2+1} + e^{-\pi s}$$

$$Y(s) = \frac{s}{(s^2+1)^2} + \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{2}\cdot\frac{2s}{(s^2+1)^2} + e^{-\pi s}\cdot\frac{1}{s^2+1} + \frac{1}{s^2+1}$$

Applying $\mathcal{L}^{-1}$:

$$y(t) = \frac{1}{2}t\sin t + \sin(t-\pi)\mathcal{U}(t-\pi) + \sin t$$

$\sin(t-\pi) = -\sin t$ by looking at graph of $f(t)=\sin t$

or by $\sin(\alpha+\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$y(t) = \frac{1}{2}t\sin t - \sin t\,\mathcal{U}(t-\pi) + \sin t$$

$$y(t) = \begin{cases} \frac{1}{2}t\sin t + \sin t, & 0 \le t < \pi \\ \frac{1}{2}t\sin t, & t \ge \pi \end{cases}$$

(34) $y''(t) + 4y(t) = f(t)$ $\quad y(0) = 0$

$y'(0) = 2$

$f(t) = 1 + \mathcal{U}(t-\pi) - 2\mathcal{U}(t-2\pi)$

$y''(t) + 4y(t) = 1 + \mathcal{U}(t-\pi) - 2\mathcal{U}(t-2\pi)$

Applying $\mathcal{L}$:

$\underline{s^2Y(s) - sy(0) - y'(0)} + 4Y(s) = \frac{1}{s} + e^{-\pi s}\cdot\frac{1}{s} + e^{-2\pi s}\cdot\frac{-2}{s}$

$(s^2+4)Y(s) = 2 + \frac{1}{s}\left[1 + e^{-\pi s} - 2e^{-2\pi s}\right]$

$Y(s) = \frac{2}{s^2+4} + \frac{1}{s(s^2+4)}\left[1 + e^{-\pi s} - 2e^{-2\pi s}\right]$

$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

$1 = A(s^2+4) + (Bs+C)s$

Sub $s=0$: $1 = 4A$ $\quad A = 1/4$

$[s^2]$: $0 = A+B$ $\quad B = -1/4$

$[s]$: $0 = C$

$Y(s) = \frac{2}{s^2+4} + \left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right)\left(1 + e^{-\pi s} - 2e^{-2\pi s}\right)$

$Y(s) = \frac{2}{s^2+4} + \left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right) + \left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right)e^{-\pi s} - 2\left(\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right)e^{-2\pi s}$

Applying $\mathcal{L}^{-1}$: $y(t) = \sin 2t + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) + \left(\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right)\mathcal{U}(t-\pi)$

$-2\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-2\pi)\right)\mathcal{U}(t-2\pi)$

→ Continued

Now $\cos 2(t-\pi) = \cos(2t-2\pi) = \cos 2t$

$\cos 2(t-2\pi) = \cos(2t-4\pi) = \cos 2t$

So
$$y(t) = \sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t$$
$$+ \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) u(t-\pi)$$
$$- \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) u(t-2\pi)$$

$$y(t) = \begin{cases} \sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t, & 0 \le t < \pi \\ \sin 2t + \frac{1}{2} - \frac{1}{2}\cos 2t, & \pi \le t < 2\pi \\ \sin 2t, & t \geq 2\pi \end{cases}$$

(35) $L\, i'(t) + R\, i(t) = V(t) \qquad i(0) = 0$

with $L = 1 \quad R = 10 \qquad V(t) = 1 - \mathcal{U}(t-2)$

$$i'(t) + 10\, i(t) = 1 - \mathcal{U}(t-2)$$

$$\mathcal{L}\{ i'(t) + 10\, i(t) \} = \mathcal{L}\{ 1 - \mathcal{U}(t-2) \}$$

$$s I(s) - i(0) + 10 I(s) = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$i(0) = 0$

$$(s+10) I(s) = \frac{1}{s}(1 - e^{-2s})$$

$$I(s) = \frac{1}{s(s+10)} (1 - e^{-2s})$$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$1 = A(s+10) + Bs$$

Subs $s = 0$: $\quad 1 = 10A \quad A = 0.1$

$s = -10$: $\quad 1 = -10B \quad B = -0.1$

$$I(s) = \left(0.1 \frac{1}{s} - 0.1 \frac{1}{s+10}\right)(1 - e^{-2s})$$

$$I(s) = \left(0.1 \frac{1}{s} - 0.1 \frac{1}{s+10}\right) - e^{-2s}\left(0.1 \frac{1}{s} - 0.1 \frac{1}{s+10}\right)$$

Applying $\mathcal{L}^{-1}$: $\quad i(t) = 0.1 - 0.1 e^{-10t} - (0.1 - 0.1 e^{-10(t-2)})\, \mathcal{U}(t-2)$

$$i(t) = \begin{cases} 0.1(1 - e^{-10t}), & 0 \le t < 2 \\ 0.1(e^{-10(t-2)} - e^{-10t}), & t \ge 2 \end{cases}$$

(36) $\mathcal{L}\{y''(t) + y(t)\} = \mathcal{L}\{e^{-t} + \delta(t-\frac{\pi}{2}) + \delta(t-\pi)\}$ $\quad y(0)=2$, $y'(0)=1$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s+1} + e^{-\frac{\pi}{2}s} + e^{-\pi s}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s+1} + e^{-\frac{\pi}{2}s} + e^{-\pi s}$$

$$(s^2+1)Y(s) = \frac{2}{s+1} + 2s + 1 + e^{-\frac{\pi}{2}s} + e^{-\pi s}$$

$$Y(s) = \frac{2}{(s+1)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2+1}\left(e^{-\frac{\pi}{2}s} + e^{-\pi s}\right)$$

$$\frac{2}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$2 = A(s^2+1) + (Bs+C)(s+1)$$

Sub $s=-1$: $\quad 2 = 2A \quad A = 1$

$[s^2]$: $\quad 0 = A + B \quad B = -1$

$[1]$: $\quad 2 = A + C \quad C = 1$

$$Y(s) = \frac{1}{s+1} + \frac{-s+1}{s^2+1} + \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s^2+1}\left(e^{-\frac{\pi}{2}s} + e^{-\pi s}\right)$$

$$Y(s) = \frac{1}{s+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1} + \frac{1}{s^2+1}\left(e^{-\frac{\pi}{2}s} + e^{-\pi s}\right)$$

Applying $\mathcal{L}^{-1}$:

$$y(t) = e^{-t} + \cos t + 2\sin t + \sin(t-\tfrac{\pi}{2})\,\mathcal{U}(t-\tfrac{\pi}{2}) + \sin(t-\pi)\,\mathcal{U}(t-\pi)$$

→ Continued

$\sin(t-\frac{\pi}{2}) = -\cos t$ from the graph of $f(t) = \sin t$

or $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(t-\pi) = -\sin t$

$$y(t) = e^{-t} + \cos t + 2\sin t - \cos t\,\mathcal{U}(t-\tfrac{\pi}{2}) - \sin t\,\mathcal{U}(t-\pi)$$

$$y(t) = \begin{cases} e^{-t} + \cos t + 2\sin t, & 0 \le t < \frac{\pi}{2} \\ e^{-t} + 2\sin t, & \frac{\pi}{2} \le t < \pi \\ e^{-t} + \sin t, & t \ge \pi \end{cases}$$

(37) $\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(t) * g(t)$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\cdot\frac{1}{(s+4)}\right\} = f(t) * g(t) \qquad f(t) = e^{3t} \quad g(t) = e^{-4t}$$

$$= \int_0^t \underset{\substack{\uparrow \\ f(\tau)}}{e^{3\tau}}\ \underset{\substack{\uparrow \\ g(t-\tau)}}{e^{-4(t-\tau)}}\, d\tau$$

$$= \int_0^t e^{7\tau - 4t}\, d\tau$$

$$= \left[\frac{1}{7} e^{7\tau - 4t}\right]_{\tau=0}^{\tau=t}$$

$$= \frac{1}{7} e^{3t} - \frac{1}{7} e^{-4t}$$

(38) $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = f(t)*g(t)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\} = f(t)*g(t) \qquad f(t)=\sin t = g(t)$

$$= \int_0^t \underbrace{\sin\tau}_{f(\tau)}\ \underbrace{\sin(t-\tau)}_{g(t-\tau)}\,d\tau$$

Let's simplify $\sin A\sin B$ before integrating:

$$\left.\begin{aligned} &\cos(A-B) = \cos A\cos B + \sin A\sin B \\ -\ &\cos(A+B) = \cos A\cos B - \sin A\sin B \end{aligned}\right] \text{ formula sheet}$$

$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$\frac{\cos(A-B)}{2} - \frac{\cos(A+B)}{2} = \sin A\sin B$

$$\Rightarrow \sin\tau\sin(t-\tau) = \frac{\cos(2\tau-t)}{2} - \frac{\cos t}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \int_0^t \left(\frac{\cos(2\tau-t)}{2} - \frac{\cos t}{2}\right)d\tau$$

$$= \left[\frac{\sin(2\tau-t)}{4} - \tau\cdot\frac{\cos t}{2}\right]_{\tau=0}^{\tau=t}$$

$$= \left[\frac{\sin t}{4} - \frac{t\cos t}{2}\right] - \left[\frac{\sin(-t)}{4} - 0\right]$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{\sin t}{4}$$

$$= \frac{\sin t - t\cos t}{2}$$