(17) $m^2+4=0$

$m=\pm 2i$ $\qquad\qquad y_c = C_1\cos 2x + C_2 \sin 2x$

$y_1=\cos 2x \qquad y_2 = \sin 2x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

$$W_1=\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin 2x \\ \sec 2x & 2\cos 2x \end{vmatrix} = -\sin 2x \sec 2x = -\tan 2x$$

$$W_2=\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec 2x \end{vmatrix} = 1$$

$u_1' = \dfrac{W_1}{W} = -\dfrac{1}{2}\tan 2x \qquad\qquad u_2' = \dfrac{W_2}{W} = \dfrac{1}{2}$

$u_1 = \int -\dfrac{1}{2}\tan 2x \, dx \qquad\qquad u_2 = \int \dfrac{1}{2} dx$

$\quad = \dfrac{1}{4}\ln|\cos 2x| \cancel{+C} \qquad\qquad = \dfrac{1}{2}x \cancel{+C}$

$y_p = u_1 y_1 + u_2 y_2$

$\quad = \dfrac{1}{4}\ln|\cos 2x|\cdot\cos 2x + \dfrac{1}{2}x\sin 2x$

$y = y_c + y_p$

$y = C_1\cos 2x + C_2\sin 2x + \dfrac{1}{4}\ln|\cos 2x|\cdot\cos 2x + \dfrac{1}{2}x\sin 2x$

(18) a) $x^2y'' - 3xy' - 5y = 0$ $(x>0)$

$(m(m-1) - 3m - 5)x^m = 0$

$m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0$

$m = -1, 5$

$y = C_1 x^{-1} + C_2 x^5$

b) $x^2y'' - 3xy' + 4y = 0$

$(m(m-1) - 3m + 4)x^m = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$

$y = C_1 x^2 + C_2 x^2 \ln x$

c) $x^2y'' - 3xy' + 13y = 0$

$(m(m-1) - 3m + 13)x^m = 0$

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{-36}}{2}$

$m = \frac{4 \pm 6i}{2}$

$m = 2 \pm 3i$

$y = x^2(C_1 \cos(3\ln x) + C_2 \sin(3\ln x))$

(19) $x^2y'' - 2xy' + 2y = 0$

$$(m(m-1) - 2m + 2)x^m = 0$$

$$m^2 - 3m + 2 = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$y_1 = x \qquad y_2 = x^2 \qquad y_c = C_1 x + C_2 x^2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2 \\ x^2e^{-x} & 2x \end{vmatrix} = -x^4e^{-x}$$

$\boxed{\text{standard Form}}$ $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x^2e^{-x} \leftarrow f(x)$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & x^2e^{-x} \end{vmatrix} = x^3e^{-x}$$

$$u_1' = \frac{W_1}{W} = -x^2e^{-x} \qquad u_2' = \frac{W_2}{W} = xe^{-x}$$

see next page $\longrightarrow$

D	I
$\oplus$ $-x^2$	e^{-x}
$\ominus$ $-2x$	$-e^{-x}$
$\oplus$ -2	e^{-x}
0	$-e^{-x}$

$$u_1 = \int -x^2 e^{-x}\,dx$$
$$= (x^2+2x+2)e^{-x} \cancel{+C}$$

D	I
$\oplus$ x	e^{-x}
$\ominus$ 1	$-e^{-x}$
0	e^{-x}

$$u_2 = \int x e^{-x}\,dx$$
$$= -(x+1)e^{-x} \cancel{+C}$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$y_p = x(x^2+2x+2)e^{-x} - x^2(x+1)e^{-x}$$
$$y_p = (x^2+2x)e^{-x}$$

$$y = y_c + y_p$$
$$y = C_1 x + C_2 x^2 + (x^2+2x)e^{-x}$$

(20) $x^2y'' - xy' + y = 0 \qquad (x > 0)$

$(m(m-1) - m + 1)x^m = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$

$y_c = C_1 x + C_2 x \ln x$

$y_1 = x \qquad y_2 = x \ln x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & 1+\ln x \end{vmatrix} = -2\ln x$$

Standard Form $\quad y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x} \leftarrow f(x)$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2$$

$u_1' = \frac{W_1}{W} = -\frac{2}{x}\ln x \qquad u_2' = \frac{W_2}{W} = \frac{2}{x}$

$\longrightarrow$

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$u_1 = \int \frac{-2}{x} \ln x \, dx$

Sub $u = \ln x$

$du = \frac{dx}{x}$

$= \int -2u \, du$

$= -u^2 \cancel{+C}$

$= -(\ln x)^2$

$u_2 = \int \frac{2}{x} dx$

$= 2 \ln x \cancel{+C}$

$y_p = u_1 y_1 + u_2 y_2$

$y_p = -x(\ln x)^2 + 2x(\ln x)^2$

$y_p = x(\ln x)^2$

$y = c_1 x + c_2 x \ln x + x(\ln x)^2$

$y(1) = 2: \quad 2 = c_1$

$y = 2x + c_2 x \ln x + x(\ln x)^2$

$y' = 2 + c_2(1 + \ln x) + \left[(\ln x)^2 + 2x(\ln x) \cdot \frac{1}{x}\right]$

$y'(1) = 3: \quad 3 = 2 + c_2$

$c_2 = 1$

$y = 2x + x \ln x + x(\ln x)^2$

(21) $m x'' + \beta x' + k x = F(t)$

m: 2kg, β: 5, k: 3N/m, $F(t)$: driving force

$$2x'' + 5x' + 3x = 4\cos 2t + 3\sin 2t$$

a) $2m^2 + 5m + 3 = 0$

$(2m+3)(m+1) = 0$

$m = -\frac{3}{2}, -1$

$x_c = C_1 e^{-3t/2} + C_2 e^{-t}$

$x_p = A\cos 2t + B\sin 2t$

$x_p' = -2A\sin 2t + 2B\cos 2t$

$x_p'' = -4A\cos 2t - 4B\sin 2t$

DE

$$2(-4A\cos 2t - 4B\sin 2t) + 5(-2A\sin 2t + 2B\cos 2t) + 3(A\cos 2t + B\sin 2t) = 4\cos 2t + 3\sin 2t$$

$$(\underbrace{-8A + 10B + 3A}_{-5A+10B})\cos 2t + (\underbrace{-8B - 10A + 3B}_{-5B-10A})\sin 2t = 4\cos 2t + 3\sin 2t$$

Matching coefficients: $\left.\begin{aligned} -5A + 10B &= 4 \\ -10A - 5B &= 3 \end{aligned}\right\}$ $A = -\frac{2}{5}$ $B = \frac{1}{5}$

$x_p = -\frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t$

$x(t) = C_1 e^{-3t/2} + C_2 e^{-t} - \frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t$

As $t \to \infty$ $x(t) \to x_p$.

b) $x_p = -\frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t$

$$A = \sqrt{\left(-\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$$

$$\tan\phi = \frac{\left(-\frac{2}{5}\right)}{\left(\frac{1}{5}\right)} = -2$$

$$\phi = -1.107 \ (+\pi\ ?)$$

$$\phi = -1.107$$

$$x_p = \frac{\sqrt{5}}{5}\sin\left(2t - 1.107\right)$$

(22) a) $$x'' + 3x' + 2x = 0$$

$x(0) = 0.5\text{m}$ $\qquad x'(0) = 0$

UNITS!

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, -2$$

$$x(t) = C_1 e^{-t} + C_2 e^{-2t}$$

$x(0) = 0.5 \qquad 0.5 = C_1 + C_2$

$x'(t) = -C_1 e^{-t} - 2C_2 e^{-2t}$

$x'(0) = 0 \qquad 0 = -C_1 - 2C_2$

$C_1 = 1$
$C_2 = -0.5$

$$x(t) = e^{-t} - \frac{1}{2} e^{-2t}$$

b) $$x'' + 3x' + 2x = F_0 \cos t \qquad (F_0 \text{ constant}).$$

$$x_c = C_1 e^{-t} + C_2 e^{-2t}$$

$$\begin{cases} x_p = A\cos t + B\sin t \\ x_p' = -A\sin t + B\cos t \\ x_p'' = -A\cos t - B\sin t \end{cases}$$

DE

$$(-A\cos t - B\sin t) + 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = F_0 \cos t$$

$$(\underbrace{-A + 3B + 2A}_{A+3B})\cos t + (\underbrace{-B - 3A + 2B}_{B-3A})\sin t = F_0 \cos t$$

$$\left.\begin{array}{l} A+3B=F_0 \\ -3A+B=0 \end{array}\right] \quad A=0.1F_0 \quad B=0.3F_0$$

$$x_p = 0.1\,F_0 \cos t + 0.3F_0 \sin t$$

$$x(t) = C_1 e^{-t} + C_2 e^{-t} + 0.1F_0 \cos t + 0.3F_0 \sin t$$

As $t \to \infty$ $\quad x(t) \to x_p$

Want $\sqrt{(0.1F_0)^2 + (0.3F_0)^2} = 1$

$$0.1\,F_0^2 = 1$$

$$F_0^2 = 10$$

$$F_0 = \pm\sqrt{10}$$

$$\boxed{F_0 = \sqrt{10}} \quad \text{(given } F_0 > 0\text{)}$$

(23) $x'' + 4x' + 5x = 0$ $x(0) = 0.5$

a) $m^2 + 4m + 5 = 0$ $x'(0) = 1$

$m = \frac{-4 \pm \sqrt{-4}}{2}$

$m = -2 \pm i$

$x(t) = e^{-2t}(C_1 \cos t + C_2 \sin t)$

$x(0) = 0.5$ $0.5 = C_1$

$x'(t) = e^{-2t}(-C_1 \sin t + C_2 \cos t) - 2e^{-2t}(C_1 \cos t + C_2 \sin t)$

$x'(0) = 1$: $1 = C_2 - 2C_1$

$C_2 = 2$

$x(t) = e^{-2t}(0.5 \cos t + 2 \sin t)$

b) Set $x(t) = 0$

$e^{-2t}(0.5 \cos t + 2 \sin t) = 0$

$0.5 \cos t + 2 \sin t = 0$

$2 \sin t = -0.5 \cos t$

$\tan t = -\frac{1}{4}$

$t \approx -0.245 + n\pi$ ← period of $\tan t$ is π

for $n = 1, 2, 3, \ldots$

(24) $x'' + 5x' + 6x = 3\sin 2t - \cos 2t$

$x(0) = -0.5m \qquad x'(0) = 1$

$\uparrow \ominus$

$\downarrow \oplus$

$$m^2 + 5m + 6 = 0$$
$$(m+2)(m+3) = 0$$
$$m = -2, -3$$
$$x_c = C_1 e^{-2t} + C_2 e^{-3t}$$

$$\text{DE} \swarrow \begin{cases} x_p = A\sin 2t + B\cos 2t \\ x_p' = 2A\cos 2t - 2B\sin 2t \\ x_p'' = -4A\sin 2t - 4B\cos 2t \end{cases}$$

$$(-4A\sin 2t - 4B\cos 2t) + 5(2A\cos 2t - 2B\sin 2t) + 6(A\sin 2t + B\cos 2t) = 3\sin 2t - \cos 2t$$

$$(\underbrace{-4A - 10B + 6A}_{2A - 10B})\sin 2t + (\underbrace{-4B + 10A + 6B}_{2B + 10A})\cos 2t = 3\sin 2t - \cos 2t$$

$$\left.\begin{array}{l} 2A - 10B = 3 \\ 10A + 2B = -1 \end{array}\right] \quad A = \frac{-1}{26} \quad B = \frac{-8}{26} = \frac{-4}{13}$$

$$x_p = \frac{-1}{26}\sin 2t - \frac{4}{13}\cos 2t$$

$$x(t) = C_1 e^{-2t} + C_2 e^{-3t} - \frac{1}{26}\sin 2t - \frac{4}{13}\cos 2t$$

$x(0) = -0.5$: $\quad -0.5 = C_1 + C_2 - \frac{4}{13} \qquad C_1 + C_2 = \frac{-5}{26}$

$$x'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} - \frac{1}{13}\cos 2t + \frac{8}{13}\sin 2t$$

$x'(0) = 1$: $\quad 1 = -2C_1 - 3C_2 - \frac{1}{13} \qquad -2C_1 - 3C_2 = \frac{28}{26}$

$$\boxed{x(t) = \frac{1}{2}e^{-2t} - \frac{9}{13}e^{-3t} - \frac{1}{26}\sin 2t - \frac{4}{13}\cos 2t} \qquad C_1 = \frac{1}{2} \quad C_2 = \frac{-9}{13}$$

(25) Let $y = \sum_{n=0}^{\infty} C_n x^n$

$$y'' - x^2 y' + 3y = 0 \qquad y(0) = 2 \quad y'(0) = -1$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n C_n x^{n-1} + 3 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=1}^{\infty} n C_n x^{n+1} + \sum_{n=0}^{\infty} 3 c_n x^n = 0$$

exponent $k = n-2$ $\qquad k = n+1 \qquad k = n$

$n = k+2 \qquad n = k-1$

start at $k = 2$

$$2C_2 + 6c_3 x + 3C_0 + 3C_1 x$$
$$+ \sum_{k=2}^{\infty} \left[(k+2)(k+1) C_{k+2} x^k - (k-1) C_{k-1} x^k + 3C_k x^k \right] = 0$$

$$(3C_0 + 2C_2) + (3C_1 + 6C_3)x + \sum_{k=2}^{\infty} \left[(k+2)(k+1) C_{k+2} - (k-1) C_{k-1} + 3C_k \right] x^k =$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$3C_0 + 2C_2 = 0 \qquad 3C_1 + 6C_3 = 0 \qquad (k+2)(k+1) C_{k+2} = (k-1) C_{k-1} - 3C_k$

$C_2 = -\frac{3}{2} C_0 \qquad C_3 = -\frac{1}{2} C_1 \qquad C_{k+2} = \dfrac{(k-1) C_{k-1} - 3C_k}{(k+2)(k+1)}$

$k = 2, 3, 4 \ldots$

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$C_2 = -\frac{3}{2} C_0$ $\quad$ $y(0)=2$ $\boxed{C_0 = 2}$ $\quad$ $\boxed{C_2 = -3}$

$C_3 = \frac{-1}{2} C_1$ $\quad$ $y'(0) = -1$ $\boxed{C_1 = -1}$ $\quad$ $\boxed{C_3 = \frac{1}{2}}$

$$C_4 = \frac{C_1 - 3C_2}{4 \cdot 3}$$

$$= \frac{-1 - 3(-3)}{12}$$

$$= \frac{2}{3}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$$

$$y = 2 - x - 3x^2 + \frac{1}{2} x^3 + \frac{2}{3} x^4 + \dots$$

$y'' - x^2 y' + 3y = 0$
has no singular points

$\Rightarrow$ The series solution converges for all x.

(26) $(x+1)y'' - (2-x)y' + y = 0 \qquad y(0)=2 \qquad y'(0)=-1$

Let $y = \sum_{n=0}^{\infty} c_n x^n$

$$(x+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (2-x)\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^n$$

$k=n-1$, $n=k+1$ $\qquad$ $k=n-2$, $n=k+2$ $\qquad$ $k=n-1$, $n=k+1$ $\qquad$ $k=n$

$$+\sum_{n=0}^{\infty} c_n x^n = 0$$

$k=n$

Start at $k=1$

$$2c_2 - 2c_1 + c_0 + \sum_{k=1}^{\infty}\left[(k+1)kc_{k+1} + (k+2)(k+1)c_{k+2} - 2(k+1)c_{k+1} + kc_k + c_k\right]x^k = 0$$

$2c_2 - 2c_1 + c_0 = 0$

$\boxed{c_2 = c_1 - \frac{1}{2}c_0}$

$$(k+2)(k+1)c_{k+2} + (k-2)(k+1)c_{k+1} + (k+1)c_k = 0$$

$$c_{k+2} = \frac{-(k-2)(k+1)c_{k+1} - (k+1)c_k}{(k+2)(k+1)}$$

$$\text{or } c_{k+2} = \frac{-(k-2)c_{k+1} - c_k}{k+2}$$

$k = 1, 2, 3, \ldots$

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$y(0)=2 \qquad C_0 = 2$

$y'(0) = -1 \qquad C_1 = -1$

$C_2 = C_1 - \frac{1}{2}C_0 = -2$

$C_3 = \dfrac{1 \cdot C_2 - C_1}{3} = \dfrac{-2-(-1)}{3} = -\dfrac{1}{3}$

$C_4 = \dfrac{0 \cdot C_3 - C_2}{4} = \dfrac{2}{4} = \dfrac{1}{2}$

$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots$

$y = 2 - x - 2x^2 - \frac{1}{3}x^3 + \frac{1}{2}x^4 + \ldots$

$$(x+1)y'' - (2-x)y' + y = 0$$

$$y'' - \frac{(2-x)}{x+1} + \frac{1}{x+1}y = 0$$

singular point : $x = -1$

The series solution converges for <u>at least</u> $|x| < 1 \leftarrow$ smallest absolute value among singular points

(27)

$$y = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$

$$y'' + e^x y = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \cdots) + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + \cdots) = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + \cdots) + c_0 + (c_0 + c_1)x + \left(c_2 + c_1 + \frac{c_0}{2}\right)x^2 + \cdots = 0$$

$$(2c_2 + c_0) + (6c_3 + c_0 + c_1)x + \left(12c_4 + c_2 + c_1 + \frac{c_0}{2}\right)x^2 + \cdots = 0$$

$\downarrow$ $\qquad\qquad$ $\downarrow$ $\qquad\qquad$ $\downarrow$

$2c_2 + c_0 = 0$

$c_2 = -\frac{1}{2} c_0$

$6c_3 + c_0 + c_1 = 0$

$c_3 = -\frac{1}{6}(c_0 + c_1)$

$12c_4 + c_2 + c_1 + \frac{c_0}{2} = 0$

$c_4 = \frac{-c_2}{12} - \frac{c_1}{12} - \frac{c_0}{24}$

To get y_1:

c_0 constant

$c_1 = 0$

$c_2 = -\frac{1}{2} c_0$

$c_3 = -\frac{1}{6}(c_0 + c_1) = -\frac{1}{6} c_0$

To get y_2:

$c_0 = 0$

c_1 constant

$c_2 = -\frac{1}{2} c_0 = 0$

$c_3 = -\frac{1}{6}(c_0 + c_1) = -\frac{1}{6} c_1$

$c_4 = \frac{-c_2}{12} - \frac{c_1}{12} - \frac{c_0}{24} = \frac{-c_1}{12}$

$$y = c_0\Big(\underbrace{1 - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots}_{y_1}\Big) + c_1\Big(\underbrace{x - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 + \cdots}_{y_2}\Big)$$