(11)

200L

$m(t)$ = mass of salt (grams)

$m(0) = 100$

Inflow: $\frac{5g}{L} \cdot \frac{4L}{min} = 20 \ g/min$

Outflow: $\frac{m \ g}{200 \ L} \cdot \frac{4L}{min} = \frac{4m}{200} \ g/min$

or $\frac{m}{50} \ g/min$

Inflow − Outflow

a)

$$\frac{dm}{dt} = 20 - \frac{m}{50}$$

$$\boxed{\frac{dm}{dt} + \frac{1}{50} m = 20}$$ Linear

$$p(t) = \frac{1}{50}$$

$$\int p(t)dt = \frac{1}{50}t$$

$$e^{\int p(t)dt} = e^{\frac{1}{50}t}$$

$$e^{\frac{1}{50}t} \frac{dm}{dt} + \frac{1}{50} e^{\frac{1}{50}t} \cdot m = 20e^{\frac{1}{50}t}$$

$$\int \left(e^{\frac{1}{50}t} \frac{dm}{dt} + \frac{1}{50} e^{\frac{1}{50}t} m\right)dt = \int 20e^{\frac{1}{50}t} dt$$

$e^{\frac{1}{50}t} m = 1000 e^{\frac{1}{50}t} + C$

$m = 1000 + Ce^{-\frac{1}{50}t}$

Sub $u = \frac{1}{50}t$

$du = \frac{1}{50}dt$

$50du = dt$

$\int 20e^{\frac{1}{50}t}\,dt$

$= \int 1000e^{u}\,du$

$= 1000e^{u} + C$

$= 1000e^{\frac{1}{50}t} + C$

$m(0)=100$: $\quad 100 = 1000 + C$

$m=100$, $t=0$ $\quad C = -900$

$$\boxed{m(t) = 1000 - 900e^{-\frac{1}{50}t}}$$

b) $\lim_{t\to\infty} m(t) = 1000 - 900(0)$

$= 1000$

c) Set $m(t) = 600$

$600 = 1000 - 900e^{-\frac{1}{50}t}$

$-400 = -900e^{-\frac{1}{50}t}$

$\frac{4}{9} = e^{-\frac{1}{50}t}$

$\ln\left(\frac{4}{9}\right) = -\frac{1}{50}t$

$t = -50\ln\left(\frac{4}{9}\right)$

or $t \approx 40.5$ minutes

(12) Newton's Law of Cooling can be described a couple of different ways:

$$\frac{dT}{dt} = k(T - T_m) \qquad k<0$$

or $$\frac{dT}{dt} = -k(T - T_e) \qquad k>0$$

where T_m or T_e represents the temperature of the surrounding medium.

a) Given $\frac{dT}{dt} = -k(T - T_e) \quad (k>0)$

$$\int \frac{dT}{T - T_e} = \int -k\,dt$$

$$\ln|T - T_e| = -kt + C_1$$

$$|T - T_e| = e^{-kt + C_1}$$

$$T - T_e = Ce^{-kt} \qquad C = \pm e^{C_1}$$

$$T = T_e + Ce^{-kt}$$

$T(0) = T_o$

$t=0$
$T=T_o$

$$T_o = T_e + C$$

$$C = T_o - T_e$$

$$\boxed{T = T_e + (T_o - T_e)e^{-kt}}$$

Note: Solving $\frac{dT}{dt} = k(T - T_m) \quad (k<0)$

gives $T = T_m + (T_o - T_m)e^{kt}$

b) Let t = # hours after 8:00pm

$T(0) = 25$ @ 8 pm

$T(2) = 15$ @ 10 pm

$T(t_1) = 37$ @ time of murder

t_1 will be negative
Murder occurred <u>before</u> 8pm

Given $T_e = 10$

$$T = 10 + (T_0 - 10)e^{-kt}$$

$T(0)=25$
or $T_0 = 25$

$$T = 10 + 15e^{-kt}$$

$T(2) = 15$

$$15 = 10 + 15e^{-2k}$$

$$5 = 15e^{-2k}$$

$$\frac{1}{3} = e^{-2k}$$

$$\ln\frac{1}{3} = -2k$$

$$k = \frac{-1}{2}\ln\frac{1}{3} = \frac{1}{2}\ln 3$$

$$T = 10 + 15e^{-\frac{1}{2}\ln 3 \cdot t}$$

$T(t_1) = 37$

$$37 = 10 + 15e^{-\frac{1}{2}\ln 3 \cdot t_1}$$

$$27 = 15e^{-\frac{1}{2}\ln 3 \cdot t_1}$$

$$\frac{27}{15} = e^{-\frac{1}{2}\ln 3 \cdot t_1}$$

$$\ln\left(\frac{27}{15}\right) = -\frac{1}{2}\ln 3 \cdot t_1$$

$$t_1 = \frac{-2\ln\left(\frac{27}{15}\right)}{\ln 3}$$

$$t_1 \approx -1.07 \text{ hours}$$

or $t_1 \approx -64$ minutes

Time of murder was $\approx$ 64 minutes before 8:00 pm

$\approx$ 6:56 pm

(13) a) $y'' + 4y = 0$

$m^2 + 4 = 0$

$m = \pm 2i$

$y_c = C_1 \cos 2x + C_2 \sin 2x$

$f(x) = 5x + \sin(2x)$ ← duplication with y_c

$y_p = Ax + B + x(C\cos 2x + D\sin 2x)$

or $y_p = Ax + B + Cx\cos 2x + Dx\sin 2x$

b) $y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$

$y_c = C_1 e^x + C_2 x e^x$

$f(x) = 3x^2 + 5xe^x$ ← duplication with y_c

$y_p = Ax^2 + Bx + C + (Dx + E)x^2 e^x$

c) $y'' + 2y' + 5y = 0$

$m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$$

$m = -1 \pm 2i$

$y_c = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$

$f(x) = 3e^{-x} + 4e^{-x} \sin 2x - 2\cos 3x$

→ duplication with y_c

$y_p = Ae^{-x} + xe^{-x}(B\cos 2x + C\sin 2x) + D\cos 3x + E\sin 3x$

(14) Given $y_1 = e^{x^2}$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$y'' - 4xy' + (4x^2 - 2)y = 0$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$P(x) = -4x$$

$$-\int P(x)dx = -(-2x^2) = 2x^2$$

$$y_2 = e^{x^2} \int \frac{e^{2x^2}}{(e^{x^2})^2}\, dx$$

$$= e^{x^2} \int \frac{e^{2x^2}}{e^{2x^2}}\, dx$$

$$= e^{x^2} \int 1\, dx$$

$$= xe^{x^2}$$

$$y = C_1 e^{x^2} + C_2 x e^{x^2}$$

$y(0) = 1$: $1 = C_1$

$$y = e^{x^2} + C_2 x e^{x^2}$$

$$y' = 2xe^{x^2} + C_2\left(e^{x^2} + x \cdot 2x e^{x^2}\right)$$

$y'(0) = 2$ $\quad 2 = C_2$

$$\boxed{y = e^{x^2} + 2xe^{x^2} \quad \text{or} \quad y = (1+2x)e^{x^2}}$$

(15) $y'' + y' - 6y = 0$

$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$y_c = C_1 e^{-3x} + C_2 e^{2x}$

$f(x) = 4e^{3x} + 5\cos 2x$

$y_p = Ae^{3x} + B\cos 2x + C\sin 2x$

$y_p' = 3Ae^{3x} - 2B\sin 2x + 2C\cos 2x$

$y_p'' = 9Ae^{3x} - 4B\cos 2x - 4C\sin 2x$

$y'' + y' - 6y = 4e^{3x} + 5\cos 2x$

$(9Ae^{3x} - 4B\cos 2x - 4C\sin 2x) + (3Ae^{3x} - 2B\sin 2x + 2C\cos 2x)$

$-6(Ae^{3x} + B\cos 2x + C\sin 2x) = 4e^{3x} + 5\cos 2x$

$6Ae^{3x} + (\underbrace{-4B + 2C - 6B}_{-10B+2C})\cos 2x + (\underbrace{-4C - 2B - 6C}_{-10C-2B})\sin 2x = 4e^{3x} + 5\cos 2x$

Matching coefficients:

$6A = 4 \qquad A = 2/3$

$-10B + 2C = 5$ (1)

$-2B - 10C = 0$ (2)

$-10B + 2C = 5$ ①

$10B + 50C = 0$ $-5 \times$ ②

$52C = 5$

$C = \frac{5}{52}$

$\rightarrow$ ① $-10B + 2 \cdot \frac{5}{52} = 5$

$-10B = \frac{250}{52}$

$B = \frac{-25}{52}$

$y_p = \frac{2}{3} e^{3x} - \frac{25}{52} \cos 2x + \frac{5}{52} \sin 2x$

$y = y_c + y_p$

$y = C_1 e^{-3x} + C_2 e^{2x} + \frac{2}{3} e^{3x} - \frac{25}{52} \cos 2x + \frac{5}{52} \sin 2x$

(16) $y'' - 5y' + 6y = 0$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

$y_c = C_1 e^{2x} + C_2 e^{3x}$

$f(x) = x + 2 + 3e^{2x}$ → duplication with y_c

$y_p = Ax + B + Cxe^{2x}$

$y_p' = A + C(x \cdot 2e^{2x} + e^{2x})$

$= A + C(2x+1)e^{2x}$

$y_p'' = C[(2x+1) \cdot 2e^{2x} + e^{2x}(2)]$

$= C(4x+4)e^{2x}$

$y'' - 5y' + 6y = x + 2 + 3e^{2x}$

$C(4x+4)e^{2x} - 5[A + C(2x+1)e^{2x}] + 6[Ax + B + Cxe^{2x}]$

$= x + 2 + 3e^{2x}$

$6Ax + (6B - 5A) + \underbrace{(4C - 10C + 6C)}_{0}xe^{2x} + \underbrace{(4C - 5C)}_{-C}e^{2x}$

$= x + 2 + 3e^{2x}$

Matching coefficients:

$6A = 1$

$A = \frac{1}{6}$

$6B - 5A = 2$

$6B - \frac{5}{6} = 2$

$6B = \frac{17}{6}$

$B = \frac{17}{36}$

$-C = 3$

$C = -3$

$y_p = \frac{1}{6}x + \frac{17}{36} - 3xe^{2x}$

$y = y_c + y_p$

$y = C_1e^{2x} + C_2e^{3x} + \frac{x}{6} + \frac{17}{36} - 3xe^{2x}$