① <u>Separable</u>

$$\int \frac{dP}{P-P^2} = \int dt$$

$$\int \frac{dP}{P(1-P)}$$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$1 = A(1-P) + BP$$

Sub $P=1$ $\quad B=1$

Sub $P=0$ $\quad A=1$

$$\int \frac{dP}{P(1-P)} = \int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP$$

$$\int \frac{1}{1-P}\, dP \quad \text{Sub } u = 1-P$$

$du = -dP$

$-du = dP$

$$= \int \frac{-du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|1-P| + C$$

$$\int \left(\frac{1}{P} - \frac{1}{1-P}\right) dP = \int dt$$

$$\ln|P| - \ln|1-P| = t + C_1$$

$$\ln\left|\frac{P}{1-P}\right| = t + C_1$$

$$\left|\frac{P}{1-P}\right| = e^{t+C_1}$$

$$\frac{P}{1-P} = Ce^t$$

$$[C = \pm e^{C_1}]$$

$P(0) = 2$:

$t = 0$

$P = 2$

$$\frac{2}{1-2} = Ce^0 \quad C = -2$$

$$\frac{P}{1-P} = -2e^t$$

$$P = -2e^t(1-P)$$

$$P = -2e^t + 2Pe^t$$

$$P - 2Pe^t = -2e^t$$

$$P(1-2e^t) = -2e^t$$

$$P = \frac{-2e^t}{1-2e^t}$$

or $\boxed{P = \dfrac{2e^t}{2e^t - 1}}$

(2) $x^3 \frac{dy}{dx} + 4x^2 y = e^{-x}$

Linear

standard Form: $\frac{dy}{dx} + \frac{4}{x} y = \frac{1}{x^3} e^{-x}$

$P(x) = \frac{4}{x}$

$\int P(x)dx = 4\ln|x| + C$

$e^{\int P(x)dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$

$(x<0)$

$x^4 \frac{dy}{dx} + 4x^3 y = x e^{-x}$

$\int \left(x^4 \frac{dy}{dx} + 4x^3 y\right) dx = \int x e^{-x} dx$

	D	I
$\oplus$	x	e^{-x}
$\ominus$	1	$-e^{-x}$
	0	e^{-x}

$\int x e^{-x} dx = -x e^{-x} - e^{-x} + C$

$x^4 y = -x e^{-x} - e^{-x} + C_1$

$y(-1) = 0$

$x = -1$

$y = 0$

$0 = e^1 - e^1 + C_1$

$C_1 = 0$

$x^4 y = -(x+1)e^{-x}$

$$\boxed{y = \frac{-(x+1)e^{-x}}{x^4}}$$

(3) $xy' + 2y = 3x$

Linear $\frac{dy}{dx} + \frac{2}{x} y = 3$

$P(x) = \frac{2}{x}$

$\int P(x)dx = 2\ln|x| + C$

$e^{\int P(x)dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2 \quad (x>0)$

$x^2 \frac{dy}{dx} + 2xy = 3x^2$

$\int \left(x^2 \frac{dy}{dx} + 2xy\right)dx = \int 3x^2 dx$

$x^2 y = x^3 + C$

$y(1)=0$

$x=1$

$y=0$

$0 = 1 + C \qquad C = -1$

$x^2 y = x^3 - 1$

$\boxed{y = x - \frac{1}{x^2}}$

(4) $\dfrac{dy}{dx} = \dfrac{y^2}{xy + x^2}$

$(xy + x^2)dy = y^2 dx$ Homogeneous of degree 2

Sub $\begin{cases} x = vy \\ dx = v\,dy + y\,dv \end{cases}$

$(vy^2 + v^2y^2)dy = y^2(v\,dy + y\,dv)$

$\underbrace{vy^2dy + v^2y^2dy = vy^2dy}_{} + y^3dv$

$v^2y^2dy = y^3dv$ (Now separable)

$\int \frac{1}{y}dy = \int \frac{1}{v^2}dv$

$\ln|y| = -v^{-1} + C$

$x = vy$

$v = \frac{x}{y}$ $\rightarrow$ $\ln|y| = -\left(\frac{x}{y}\right)^{-1} + C$

$\ln|y| = -\left(\frac{y}{x}\right) + C$

$x\ln|y| = -y + Cx$

$\boxed{x\ln|y| + y = Cx}$

⑤

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}$$

$$(3x+y)dy = (x+3y)dx$$

Homogeneous of degree 1

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$$(3x+ux)(u\,dx + x\,du) = (x+3ux)dx$$

$$3ux\,dx + 3x^2du + u^2x\,dx + ux^2du = x\,dx + 3ux\,dx$$

$$3x^2du + ux^2du = -u^2x\,dx + x\,dx$$

(Separable)

$$x^2(3+u)\,du = (1-u^2)x\,dx$$

$$\int \frac{3+u}{1-u^2}\,du = \int \frac{1}{x}\,dx$$

$$\frac{3+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$3+u = A(1+u) + B(1-u)$$

Sub $u=1$ $\quad A = 2$

Sub $u=-1$ $\quad B = 1$

$$\int \frac{3+u}{1-u^2}\,du = \int\left(\frac{2}{1-u} + \frac{1}{1+u}\right)du$$

$$\int\left(\frac{2}{1-u} + \frac{1}{1+u}\right)du = \int \frac{1}{x}\,dx$$

$$-2\ln|1-u| + \ln|1+u| = \ln|x| + C$$

$$\ln\left|\frac{1}{(1-u)^2}\right| + \ln|1+u| = \ln|x| + C$$

$$\ln\left|\frac{1+u}{(1-u)^2}\right| = \ln|x| + C$$

$y = ux$

$u = \frac{y}{x}$

$$\ln\left|\frac{1+\frac{y}{x}}{\left(1-\frac{y}{x}\right)^2}\right| = \ln|x| + C$$

(6) $\frac{dy}{dx} = y(xy^3-1)$

$$\frac{dy}{dx} = xy^4 - y$$

$$\frac{dy}{dx} + y = xy^4$$

Bernoulli
$n=4$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dx} + u^{-1/3} = x\,u^{-4/3}$$

$\begin{cases} \text{Sub } y = u^{\frac{1}{1-n}} = u^{-\frac{1}{3}} \\ \frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx} \end{cases}$

$\times(-3u^{4/3})$ $\quad \frac{du}{dx} - 3u = -3x$ Now linear

$$P(x) = -3$$

$$e^{\int P(x)dx} = e^{-3x}$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x}u = -3xe^{-3x}$$

$$\int\left(e^{-3x}\frac{du}{dx} - 3e^{-3x}u\right)dx = \int -3xe^{-3x}dx$$

	D	I
$+$	$-3x$	e^{-3x}
$-$	-3	$-\frac{1}{3}e^{-3x}$
	0	$\frac{1}{9}e^{-3x}$

$$\int -3xe^{-3x}dx = xe^{-3x} + \frac{1}{3}e^{-3x} + C_1$$

$$e^{-3x}u = xe^{-3x} + \frac{1}{3}e^{-3x} + C_1$$

$$u = x + \frac{1}{3} + C_1e^{3x}$$

$y = u^{-1/3}$
$y^{-3} = u$

$$y^{-3} = x + \frac{1}{3} + C_1e^{3x}$$

$$y^3 = \frac{1}{x + \frac{1}{3} + C_1e^{3x}}$$

$$y^3 = \frac{3}{3x+1+Ce^{3x}}$$

$$y = \sqrt[3]{\frac{3}{3x+1+Ce^{3x}}}$$

(7) $M_y = 2y\cos x - 3x^2$ $\quad N_x = ky\cos x - 3x^2$

$M_y = N_x$ with $\boxed{k=2}$

$(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + 4y)dy = 0$
is exact

$$f(x,y) = \int (y^2\cos x - 3x^2y - 2x)dx$$
$$= y^2\sin x - x^3y - x^2 + g(y)$$

<u>and</u>

$$f(x,y) = \int (2y\sin x - x^3 + 4y)dy$$
$$= y^2\sin x - x^3y + 2y^2 + h(x)$$

$$f(x,y) = y^2\sin x - x^3y + 2y^2 - x^2$$
$$\boxed{y^2\sin x - x^3y + 2y^2 - x^2 = C}$$

(8) $$\frac{dy}{dx} = \frac{1-x-y}{x+y}$$

$$\frac{dy}{dx} = \frac{1-(x+y)}{x+y}$$

Sub $u = x+y$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\leftarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \frac{1-u}{u}$$ Separable

$$\frac{du}{dx} = \frac{1-u}{u} + 1$$

$$\frac{du}{dx} = \frac{1}{u}$$

$$\int u\,du = \int dx$$

$$\frac{1}{2}u^2 = x + C_1$$

$$u^2 = 2x + C_2$$

$u = x+y$ $\boxed{(x+y)^2 = 2x + C}$

Alternatively: the DE is exact

(9) $x^2 y' = xy + y^4$

$$\frac{dy}{dx} = \frac{1}{x} y + \frac{1}{x^2} y^4$$

$$\frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x^2} y^4$$

Bernoulli $n=4$

Sub $y = u^{\frac{1}{1-n}} = u^{-1/3}$

$$\frac{dy}{dx} = -\frac{1}{3} u^{-4/3} \frac{du}{dx}$$

$$-\frac{1}{3} u^{-4/3} \frac{du}{dx} - \frac{1}{x} u^{-1/3} = \frac{1}{x^2} u^{-4/3}$$

$\times(-3u^{4/3})$ $\quad \frac{du}{dx} + \frac{3}{x} u = \frac{-3}{x^2}$ $\quad$ Linear

$$P(x) = \frac{3}{x}$$

$$\int P(x)dx = 3\ln|x| = \ln x^3 \quad (x>0)$$

$$e^{\int P(x)dx} = x^3$$

$$x^3 \frac{du}{dx} + 3x^2 u = -3x$$

$$\int\left(x^3 \frac{du}{dx} + 3x^2 u\right)dx = \int -3x\,dx$$

$$x^3 u = -\frac{3}{2} x^2 + C_1$$

$$u = -\frac{3}{2} \cdot \frac{1}{x} + \frac{C_1}{x^3}$$

$y = u^{-1/3}$

$y^{-3} = u$

$$y^{-3} = \frac{-3}{2x} + \frac{C}{x^3}$$

$$\boxed{\frac{1}{y^3} = \frac{-3}{2x} + \frac{C}{x^3}}$$

(10) $(1+xye^x)dx + (xe^x + x\cos y)dy = 0$

$M_y = xe^x \qquad N_x = (xe^x + e^x) + \cos y$

Not Exact

But $\dfrac{M_y - N_x}{N} = \dfrac{-e^x - \cos y}{xe^x + x\cos y} = \dfrac{-1}{x}$ ← function of x

$\int \frac{M_y - N_x}{N}\,dx = \int \frac{-1}{x}\,dx = -\ln|x| = \ln\frac{1}{|x|} = \ln\frac{1}{x} \quad (x>0)$

$e^{\int \frac{M_y - N_x}{N}dx} = \frac{1}{x}$

$\boxed{\left(\frac{1}{x} + ye^x\right)dx + (e^x + \cos y)\,dy = 0}$

now exact

$f(x,y) = \int\left(\frac{1}{x} + ye^x\right)dx$

$= \ln|x| + ye^x + g(y)$

and $f(x,y) = \int(e^x + \cos y)\,dy$

$= ye^x + \sin y + h(x)$

$f(x,y) = ye^x + \ln|x| + \sin y$

$\boxed{ye^x + \ln|x| + \sin y = C}$