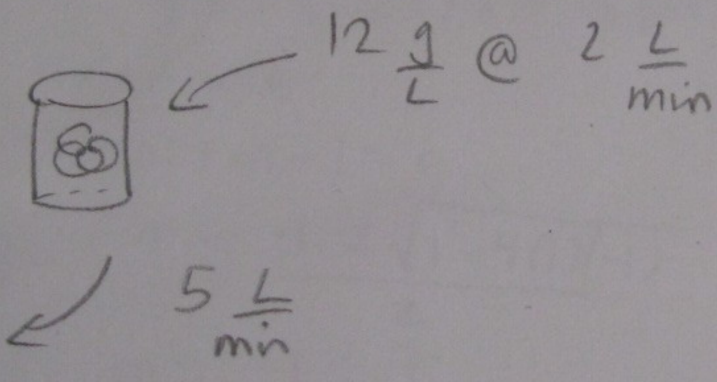
1. [5 marks]

a) The rate of change of z with respect to t (in units/year) is determined by two processes. One process decreases z by 100 units/year. The other process increases z by $0.07z$ units/year. Write down the DE but **do not solve the DE**.

$$\frac{dz}{dt} = -100 + 0.07z$$

or $$\frac{dz}{dt} = 0.07z - 100$$

b) A large tank with a capacity of 500 L initially contains 130 L of pure water. A 12 g/L sugar solution is pumped in at a rate of 2 L/min. The well-mixed solution is pumped out at a rate of 5 L/min. Let A represent the number of grams of sugar in the tank after t minutes. Write down the DE but **do not solve the DE**.



Volume $V = 130 - 3t$

Inflow Rate $= 12 \frac{g}{L} \cdot 2 \frac{L}{min} = 24 \frac{g}{min}$

Outflow Rate $= \frac{A}{130-3t} \frac{g}{L} \cdot 5 \frac{L}{min} = \frac{5A}{130-3t} \frac{g}{min}$

$$\boxed{\frac{dA}{dt} = 24 - \frac{5A}{130-3t}}$$

2. [6 marks] Solve:

a) $y'' + 10y' + 25y = 0$

$m^2 + 10m + 25 = 0$

$(m+5)^2 = 0$

$m = -5, -5$

$y = C_1 e^{-5x} + C_2 x e^{-5x}$

b) $y'' + 49y = 0$

$m^2 + 49 = 0$

$m^2 = -49$

$m = \pm\sqrt{-49}$

$m = \pm 7i$

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$y = C_1 \cos 7x + C_2 \sin 7x$

c) $y'' + y' - y = 0$

$m^2 + m - 1 = 0$

$m = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$

$m = \frac{-1 \pm \sqrt{5}}{2}$

$m = \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$ Distinct Real Roots

$y = C_1 e^{\frac{-1+\sqrt{5}}{2}x} + C_2 e^{\frac{-1-\sqrt{5}}{2}x}$

3. [4 marks] Find y_2 given that $y_1 = x^{-17/3}$ is a solution:
$21x^2y'' + 197xy' + 323y = 0$

Standard Form $\quad y'' + \frac{197}{21x} y' + \frac{323}{21x^2} y = 0$

$$P(x) = \frac{197}{21x}$$

$$\int P(x)dx = \frac{197}{21} \ln|x|$$

$$-\int P(x)dx = \frac{-197}{21} \ln|x|$$

$$= \frac{-197}{21} \ln x \qquad (x>0)$$

$$e^{-\int P(x)dx} = e^{\frac{-197}{21}\ln x}$$

$$= e^{\ln x^{\frac{-197}{21}}}$$

$$= x^{\frac{-197}{21}}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$= x^{-17/3} \int \frac{x^{-197/21}}{x^{-34/3}} dx \qquad \boxed{\frac{-197}{21} + \frac{34}{3}}$$

$$= x^{-17/3} \int x^{41/21} dx$$

$\rightarrow$

$$= x^{-17/3}\left[\frac{21}{62}x^{\frac{62}{21}}\right]$$

$$= \frac{21}{62}x^{-57/21}$$

or $\quad y_2 = \frac{21}{62}x^{-19/7}$

or $\quad y_2 = x^{-19/7}$

4. [5 marks] Solve $y'' - 6y' + 8y = e^{2x} + 5x$

y_c :

$$m^2 - 6m + 8 = 0$$
$$(m-2)(m-4) = 0$$
$$m = 2, 4$$
$$y_c = C_1 e^{2x} + C_2 e^{4x}$$

$$y_p = Ae^{2x} + Bx + C$$

bad case

$$\boxed{y_p = Axe^{2x} + Bx + C}$$

$$y_p' = 2Axe^{2x} + Ae^{2x} + B$$

$$y_p'' = 4Axe^{2x} + \underbrace{2Ae^{2x} + 2Ae^{2x}}_{4Ae^{2x}}$$

$y_p \to DE$:

$$y'' - 6y' + 8y = e^{2x} + 5x$$

$$4Axe^{2x} + 4Ae^{2x} - 6(2Axe^{2x} + Ae^{2x} + B)$$
$$+ 8(Axe^{2x} + Bx + C) = e^{2x} + 5x$$

$$0Axe^{2x} - 2Ae^{2x} + 8Bx + 8C - 6B = 1e^{2x} + 5x + 0$$

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$
$$8B = 5 \Rightarrow B = \frac{5}{8}$$
$$8C - 6B = 0 \Rightarrow 8C - \frac{30}{8} = 0 \Rightarrow C = \frac{30}{64} = \frac{15}{32}$$

$\rightarrow$

$y_p = Axe^{2x} + Bx + C$

$y_p = \frac{-1}{2}xe^{2x} + \frac{5}{8}x + \frac{15}{32}$

$y = y_c + y_p$

$y = C_1e^{2x} + C_2e^{4x} - \frac{1}{2}xe^{2x} + \frac{5}{8}x + \frac{15}{32}$