8.3.2 Variation of Parameters

Ex: Solve $\vec{x}' = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$

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1) Find $\vec{x}_c$ (solution to $\vec{x}' = A\vec{x}$)

$\Phi = [\vec{x}_1 \; \vec{x}_2]$

2) Find Φ^{-1}

3) $\vec{u} = \int \Phi^{-1}\vec{F}\, dt$

4) $\vec{x}_p = \Phi\vec{u}$

5) $\vec{x} = \vec{x}_c + \vec{x}_p$

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1) $\vec{x}_c$: Solve $\vec{x}' = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}\vec{x}$

$\lambda = 1, 4$

$\vdots$

$$\vec{x}_c = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + C_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{4t}$$

$\vec{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ $\quad$ $\vec{x}_2 = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$

$$\Phi = \begin{bmatrix} e^t & e^{4t} \\ e^t & -2e^{4t} \end{bmatrix}$$

2) Find $\underline{\Phi}^{-1}$

Recall: $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$|\underline{\Phi}| = -2e^{5t} - e^{5t}$$
$$= -3e^{5t}$$

$$\underline{\Phi}^{-1} = \frac{1}{-3e^{5t}}\begin{bmatrix} -2e^{4t} & -e^{4t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$= \frac{-1}{3}\begin{bmatrix} -2e^{-t} & -e^{-t} \\ -e^{-4t} & e^{-4t} \end{bmatrix}$$

3) $$\vec{u} = \int \underline{\Phi}^{-1}\vec{F}\,dt$$

$$= -\frac{1}{3}\int \begin{bmatrix} -2e^{-t} & -e^{-t} \\ -e^{-4t} & e^{-4t} \end{bmatrix}\begin{bmatrix} 0 \\ e^{3t} \end{bmatrix} dt$$

$$= -\frac{1}{3}\int \begin{bmatrix} -e^{2t} \\ e^{-t} \end{bmatrix} dt$$

$$= -\frac{1}{3}\begin{bmatrix} -\frac{1}{2}e^{2t} \\ -e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}e^{2t} \\ \frac{1}{3}e^{-t} \end{bmatrix}$$

4) $\vec{x}_p = \Phi \vec{u}$

$$= \begin{bmatrix} e^t & e^{4t} \\ e^t & -2e^{4t} \end{bmatrix} \begin{bmatrix} \frac{1}{6}e^{2t} \\ \frac{1}{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}e^{3t} + \frac{1}{3}e^{3t} \\ \frac{1}{6}e^{3t} - \frac{2}{3}e^{3t} \end{bmatrix}$$

$$\begin{cases} \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \\ \frac{1}{6} - \frac{2}{3} = \frac{-1}{2} \end{cases}$$

$$= \begin{bmatrix} \frac{1}{2}e^{3t} \\ \frac{-1}{2}e^{3t} \end{bmatrix}$$

5) $\vec{x} = \vec{x}_c + \vec{x}_p$

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{4t} + \begin{bmatrix} \frac{1}{2}e^{3t} \\ \frac{-1}{2}e^{3t} \end{bmatrix}$$

Ex: Solve $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix}$

1) $\vec{x}_c$: Solve $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{x}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + 1 = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\lambda = \frac{2 \pm 2i}{2}$$

$$\lambda = 1 \pm i$$

$\lambda = 1+i$: $\quad [A - (1+i)I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 1-(1+i) & -1 & 0 \\ 1 & 1-(1+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$\boxed{x_2 = a}$

$x_1 - ix_2 = 0 \quad \boxed{x_1 = ai}$

$$\bar{x} = \begin{bmatrix} i \\ 1 \end{bmatrix} a$$

$$\bar{K}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{B}_1 = Re(\bar{K}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \bar{B}_2 = Im(\bar{K}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{X}_1 = e^{\alpha t}(\bar{B}_1 \cos\beta t - \bar{B}_2 \sin\beta t)$$

$$\bar{X}_1 = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right)$$

$$\boxed{\bar{X}_1 = e^{t}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}$$

$$\bar{X}_2 = e^{\alpha t}(\bar{B}_1 \sin\beta t + \bar{B}_2 \cos\beta t)$$

$$\bar{X}_2 = e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right)$$

$$\boxed{\bar{X}_2 = e^{t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}$$

$$\bar{X}_c = C_1\begin{bmatrix} -e^{t}\sin t \\ e^{t}\cos t \end{bmatrix} + C_2\begin{bmatrix} e^{t}\cos t \\ e^{t}\sin t \end{bmatrix}$$

$$\bar{\Phi} = \begin{bmatrix} -e^{t}\sin t & e^{t}\cos t \\ e^{t}\cos t & e^{t}\sin t \end{bmatrix}$$

2) $\underline{\Phi}^{-1}$:

$$|\underline{\Phi}| = -e^{2t}\sin^2 t - e^{2t}\cos^2 t$$
$$= -e^{2t}(\sin^2 t + \cos^2 t)$$
$$= -e^{2t}$$

$$\underline{\Phi}^{-1} = \frac{1}{-e^{2t}}\begin{bmatrix} e^t\sin t & -e^t\cos t \\ -e^t\cos t & -e^t\sin t \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$

3) $\vec{u} = \int \underline{\Phi}^{-1}\vec{F}\,dt$

$$= \int \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}\begin{bmatrix} e^t\cos t \\ e^t\sin t \end{bmatrix} dt$$

$$= \int \begin{bmatrix} -\sin t\cos t + \cos t\sin t \\ \cos^2 t + \sin^2 t \end{bmatrix} dt$$

$$= \int \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} 0 \\ t \end{bmatrix}$$

4) $\vec{x}_p = \underline{\Phi}\,\vec{u}$

$$= \begin{bmatrix} -e^t \sin t & e^t \cos t \\ e^t \cos t & e^t \sin t \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} te^t \cos t \\ te^t \sin t \end{bmatrix}$$

5) $\vec{x} = \vec{x}_c + \vec{x}_p$

$$\vec{x} = C_1 \begin{bmatrix} -e^t \sin t \\ e^t \cos t \end{bmatrix} + C_2 \begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix} + \begin{bmatrix} te^t \cos t \\ te^t \sin t \end{bmatrix}$$