8.1 Linear Systems of DE's

Ex:
$$\begin{cases} \frac{dx}{dt} = x - 3y + t \\ \frac{dy}{dt} = x - y + 1 \end{cases}$$

in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

symbolically:

$$\vec{X}' = A\vec{X} + \vec{F}$$

System is homogeneous if $\vec{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Solution to system:

$$\vec{X} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \begin{matrix} \leftarrow x \text{ (a function of } t) \\ \leftarrow y \text{ (a function of } t) \end{matrix}$$

Ex: Check that $\vec{x} = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$

solves $\vec{x}' = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \vec{x}$

$$\begin{cases} \vec{x} = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} \\ \vec{x}' = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix} \end{cases}$$

$$LS = \vec{x}' = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$$

$$RS = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix} \checkmark$$

Preview

Notice $A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 4$

Solution $\vec{x} = \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$

Ex: Given general solution

$$\vec{x} = C_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + C_2 \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix},$$

Solve if $\vec{x}(0) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$$\vec{x} = C_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + C_2 \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$$

Sub $t=0$ $\vec{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$: $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{cases} C_1 + C_2 = 6 & (1) \\ C_1 - 2C_2 = 3 & (2) \end{cases}$$

(1) - (2) : $3C_2 = 3$

$$\boxed{C_2 = 1}$$

$C_2 = 1 \rightarrow$ (1) : $\boxed{C_1 = 5}$

$$\vec{x} = 5 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix}$$