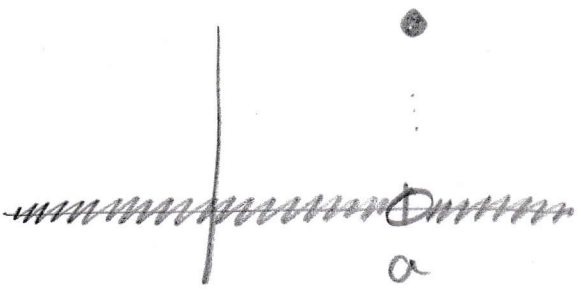
7.5 The Dirac Delta

Let $a \geq 0$

The Dirac Delta is

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t \neq a \end{cases}$$



$\delta(t-a)$ is a "generalized function" or "distribution"

Models : impulses in signal processing
point masses
point charges

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

Ex: 1 kg mass at rest at equilibrium position.
No damping.
Spring constant = 1 N/m.
Mass is struck at $t=\pi$ seconds,
Corresponding to an external force $f(t)=\delta(t-\pi)$
Find $x(t)$ using $\mathcal{L}$.

Spring-Mass System:

$$mx'' + \beta x' + kx = f(t)$$

$$1x'' + 0x' + 1x = \delta(t-\pi)$$

$$x'' + x = \delta(t-\pi) \qquad x(0)=0,\ x'(0)=0$$

1) Apply $\mathcal{L}$:

$$\left[s^2X(s) - sx(0) - x'(0)\right] + X(s) = e^{-\pi s}$$

$x(0)=0$, $x'(0)=0$:

$$s^2X(s) + X(s) = e^{-\pi s}$$

2) Solve for $X(s)$:

$$(s^2+1)X(s) = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2+1}$$

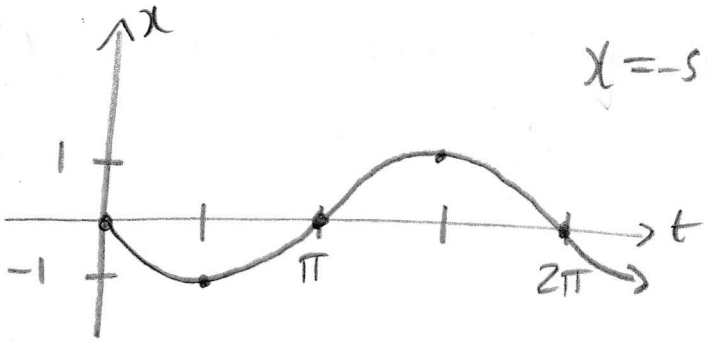
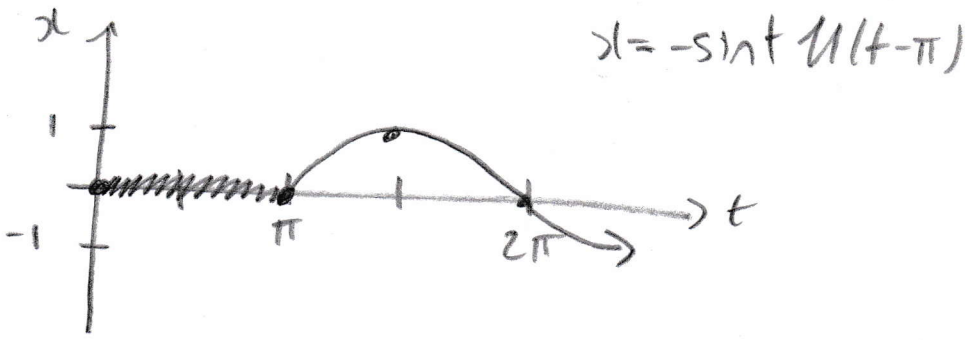
3) Apply $\mathcal{L}^{-1}$:

$$x(t) = \mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{1}{s^2+1}\right\}$$

$$= f(t-\pi)\,u(t-\pi) \longrightarrow F(s) = \frac{1}{s^2+1}$$

$$f(t) = \sin t$$

$$= \sin(t-\pi)\,u(t-\pi)$$

$$\sin(t-\pi) = \sin t \cos\pi - \cos t \sin\pi = -\sin t$$

$$= -\sin t\, u(t-\pi)$$