7.4 Convolutions and Periodic Functions

See formula sheet for transformations of:

$t\sin\omega t$

$t\cos\omega t$

$\dfrac{\sin\omega t - \omega t\cos\omega t}{2\omega^3}$

Ex: Solve using $\mathcal{L}$:

$$x'' + 9x = \cos 3t$$

$$x(0)=0,\quad x'(0)=1$$

1) Apply $\mathcal{L}$:

$$[s^2X(s) - sx(0) - x'(0)] + 9X(s) = \frac{s}{s^2+9}$$

$x(0)=0$, $x'(0)=1$:

$$s^2X(s) - 1 + 9X(s) = \frac{s}{s^2+9}$$

2) Solve for $X(s)$:

$$(s^2+9)X(s) = 1 + \frac{s}{s^2+9}$$

$$X(s) = \frac{1}{s^2+9} + \frac{s}{(s^2+9)^2}$$

3) Apply $\mathcal{L}^{-1}$:

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{s^2+9} + \frac{1}{6}\cdot\frac{6s}{(s^2+9)^2}\right\}$$

$$x(t) = \frac{1}{3}\sin 3t + \frac{1}{6}t\sin 3t$$

Convolution of f and g:

$$f * g = \int_0^t f(\theta)\, g(t-\theta)\, d\theta$$

Note: Text writes

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s)$$

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f * g$$

Ex: Find $e^t * \cos t$

Method I. $e^t * \cos t = \int_0^t e^{\theta} \cos(t-\theta)\, d\theta$

Long integration by parts ☹

Method II. $\mathcal{L}\{e^t * \cos t\} = F(s) \cdot G(s) \rightarrow$

$f(t) = e^t$, $F(s) = \frac{1}{s-1}$

$g(t) = \cos t$, $G(s) = \frac{s}{s^2+1}$

$$= \frac{1}{s-1} \cdot \frac{s}{s^2+1}$$

$$= \frac{s}{(s-1)(s^2+1)}$$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$= \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \qquad \text{(Check)}$$

Now $e^t * \cos t = \mathcal{L}^{-1}\left\{ \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \right\}$

$$e^t * \cos t = \frac{1}{2}e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

Ex: Find $\mathcal{L}^{-1}\left\{ \frac{2}{s(s^2+4)} \right\}$ using a Convolution

$$= \mathcal{L}^{-1}\left\{ \frac{2}{s^2+4} \cdot \frac{1}{s} \right\}$$

$$= f * g$$

$$= \int_0^t f(\theta) g(t-\theta)\, d\theta$$

$\rightarrow$

$F(s) = \frac{2}{s^2+4}$

$f(t) = \sin 2t$

$G(s) = \frac{1}{s}$

$g(t) = 1$

$$= \int_0^t \sin 2\theta \,(1)\, d\theta$$

$$= \left[-\frac{1}{2}\cos 2\theta \right]_{\theta=0}^{\theta=t}$$

$$= -\frac{1}{2}\cos 2t + \frac{1}{2}$$

Ex: Solve for $f(t)$:

$$f(t) = 6t - \int_0^t f(\theta) e^{t-\theta} \, d\theta$$

Apply $\mathcal{L}$:

$$F(s) = \frac{6}{s^2} - \mathcal{L}\{f * e^t\}$$

$$F(s) = \frac{6}{s^2} - F(s) \cdot \frac{1}{s-1}$$

Solve for $F(s)$:

$$F(s) \cdot \frac{1}{s-1} + F(s) = \frac{6}{s^2}$$

$$\left[\frac{1}{s-1} + 1\right] F(s) = \frac{6}{s^2}$$

$$\frac{s}{s-1} F(s) = \frac{6}{s^2}$$

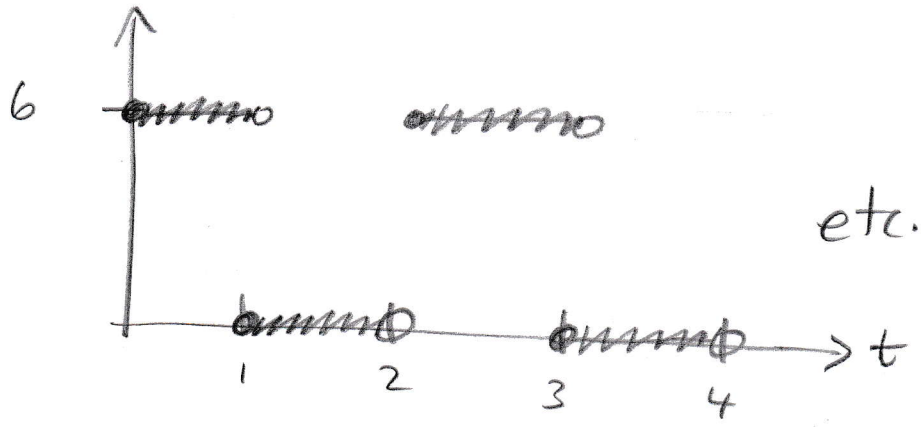
$$F(s) = \frac{6(s-1)}{s^3}$$

Apply $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^2} - \frac{6}{s^3}\right\}$$

$$f(t) = 6t - 3t^2$$

Periodic Functions



$f(t)$ is periodic with period $T=2$

If $f(t)$ has period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Ex: Find $\mathcal{L}\{f(t)\}$ for $f(t)$ shown above. Simplify your answer.

$T=2$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 6e^{-st}\,dt + \cancel{\int_1^2 0\,dt} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{6}{s} e^{-st} \right]_{t=0}^{t=1}$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{6}{s} e^{-s} + \frac{6}{s} \right]$$

$$= \frac{1}{1-(e^{-s})^2} \cdot \frac{6}{s} \left[1 - e^{-s} \right]$$

$$= \frac{6}{s} \cdot \frac{(1-e^{-s})}{(1-e^{-s})(1+e^{-s})}$$

$$= \frac{6}{s(1+e^{-s})}$$