7.3 Translations

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Ex: Evaluate:

a) $\mathcal{L}\{e^{-3t} t^5\}$

$= F(s+3)$ $\longrightarrow$ $f(t) = t^5$, $F(s) = \frac{5!}{s^6}$

$= \frac{5!}{(s+3)^6}$

b) $\mathcal{L}\{e^{4t} t\sin 6t\}$

$= F(s-4)$ $\longrightarrow$ $f(t) = t\sin 6t$, $F(s) = \frac{12s}{(s^2+36)^2}$

$= \frac{12(s-4)}{((s-4)^2+36)^2}$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Ex: $\mathcal{L}^{-1}\left\{\frac{-6}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{(s+2)^3}\right\}$

$= \mathcal{L}^{-1}\{F(s+2)\}$

$= e^{-2t} f(t) \longrightarrow F(s) = \frac{-6}{s} + \frac{2}{s^2} + \frac{1}{s^3}$

$= -6\cdot\frac{1}{s} + 2\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{2}{s^3}$

$= e^{-2t}\left(-6 + 2t + \frac{t^2}{2}\right)$ $\quad f(t) = -6 + 2t + \frac{t^2}{2}$

Ex: $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+6s+13}\right\}$

$s^2+6s+13$ doesn't factor. Complete the square.

$\frac{6}{2}=3$

$$s^2+6s+13 = (s+3)^2+?$$
$$= (s+3)^2+4$$
$$= (s+3)^2+2^2$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+6s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+1}{(s+3)^2+2^2}\right\}$$

Want $2s+1 = ?(s+3)+?$
$= 2(s+3)+?$
$= 2(s+3)-5$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+3)-5}{(s+3)^2+2^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+2^2}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+2^2}\right\}$$

$$= 2e^{-3t}\cos 2t - \frac{5}{2}e^{-3t}\sin 2t$$

Ex: Solve using the Laplace Transform

$y'' + 4y' + 20y = 50e^t$, $y(0) = 9$, $y'(0) = -28$

1) Apply $\mathcal{L}$:

$$[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 20Y(s) = \frac{50}{s-1}$$

$$s^2Y(s) - 9s + 28 + 4sY(s) - 36 + 20Y(s) = \frac{50}{s-1}$$

2) Solve for $Y(s)$:

$$(s^2 + 4s + 20)Y(s) = 9s + 8 + \frac{50}{s-1}$$

$$Y(s) = \underbrace{\frac{9s+8}{s^2+4s+20} + \frac{50}{(s-1)(s^2+4s+20)}}_{\text{Combine into 1 fraction}}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{(9s+8)(s-1) + 50}{(s-1)(s^2+4s+20)}\right\}. \quad (\star)$$

Partial Fractions:

$$\frac{9s^2 - s + 42}{(s-1)(s^2+4s+20)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4s+20}$$

$$9s^2 - s + 42 = A(s^2+4s+20) + (Bs+C)(s-1)$$

Sub $s=1$: So $= 25A$

$$\boxed{A=2}$$

Match s^2 terms:

$$9s^2 = As^2 + Bs^2$$

$$9 = A + B$$

$$\boxed{B=7}$$

Match constants:

$$42 = 20A - C$$

$$\boxed{C=-2}$$

From ~~A~~: $y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s-1} + \frac{7s-2}{s^2+4s+20} \right\}$

$$s^2+4s+20 = (s+2)^2 + ?$$
$$= (s+2)^2 + 16$$
$$= (s+2)^2 + 4^2$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s-1} + \frac{7s-2}{(s+2)^2+4^2} \right\}$$

Want $7s-2 = ?(s+2)+?$

$= 7(s+2)-16$

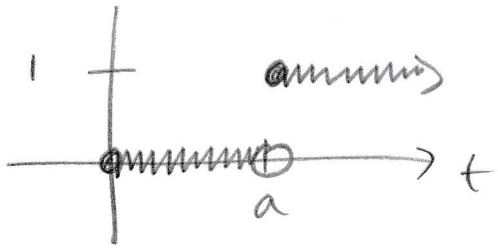
$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-1} + \frac{7(s+2)-16}{(s+2)^2+4^2}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-1} + 7\cdot\frac{s+2}{(s+2)^2+4^2} - 4\cdot\frac{4}{(s+2)^2+4^2}\right\}$$

$$y(t) = 2e^{t} + 7e^{-2t}\cos 4t - 4e^{-2t}\sin 4t$$

Heaviside Function

$$u(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$



Ex: Write in terms of $u(t-a)$

a) $$f(t)=\begin{cases}0, & 0\le t<1\\ t^2, & t\ge 1\end{cases}$$

$$f(t)=t^2u(t-1)$$

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b) $$f(t)=\begin{cases}t^2, & 0\le t<2\\ 0, & t\ge 2\end{cases}$$

$$f(t)=t^2-t^2u(t-2)$$

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c) $$f(t)=\begin{cases}\frac{1}{3}, & 0\le t<2\\ 3, & 2\le t<5\\ 0, & t\ge 5\end{cases}$$

$$f(t)=1+2u(t-2)-3u(t-5)$$

$$\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

Ex: a) $\mathcal{L}\{u(t-6)\}$

$= e^{-6s}\,\mathcal{L}\{f(t+6)\}$ → $f(t)=1$, $f(t+6)=1$

$= e^{-6s}\,\mathcal{L}\{1\}$

$= \frac{e^{-6s}}{s}$

b) $\mathcal{L}\{t^2\,u(t-1)\}$

$= e^{-s}\,\mathcal{L}\{f(t+1)\}$ → $f(t)=t^2$, $f(t+1)=(t+1)^2$

$= e^{-s}\,\mathcal{L}\{(t+1)^2\}$

$= e^{-s}\,\mathcal{L}\{t^2+2t+1\}$

$= e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

c) $\mathcal{L}^{-1}\left\{\frac{1}{s-2}e^{-3s}\right\}$

$= f(t-3)\,u(t-3) \rightarrow F(s)=\frac{1}{s-2}$

$f(t)=e^{2t}$

$f(t-3)=e^{2(t-3)}$

$= e^{2(t-3)}u(t-3)$

Ex: Solve using the Laplace transform

$y'+5y=f(t)$, $y(0)=0$

where $f(t)=\begin{cases}0, & 0\le t<3\\ 2, & t\ge 3\end{cases}$

$\rightarrow$ $\boxed{y'+5y=2u(t-3),\ y(0)=0}$

1) Apply $\mathcal{L}$:

$sY(s)-y(0)+5Y(s)=\mathcal{L}\{2u(t-3)\}$

$\mathcal{L}\{2u(t-3)\}=e^{-3s}\mathcal{L}\{f(t+3)\} \rightarrow f(t)=2,\ f(t+3)=2$

$=e^{-3s}\mathcal{L}\{2\}$

$=e^{-3s}\cdot\frac{2}{s}$

$sY(s)-y(0)+5Y(s)=e^{-3s}\cdot\frac{2}{s}$

$y(0)=0$: $(s+5)Y(s) = \frac{2}{s}e^{-3s}$

2) Solve for $Y(s)$:

$$Y(s) = \frac{2}{s(s+5)}e^{-3s}$$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s(s+5)}e^{-3s}\right\} \text{ (*)}$$

Partial Fractions: $\frac{2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$

Do not include e^{-3s} in the partial fractions.

$$\boxed{2 = A(s+5) + Bs}$$

Sub $s=0$: $2 = 5A$

$$\boxed{A = 2/5}$$

Sub $s=-5$: $2 = -5B$

$$\boxed{B = -2/5}$$

From (*): $y(t) = \mathcal{L}^{-1}\left\{\left(\frac{2}{5}\cdot\frac{1}{s} - \frac{2}{5}\cdot\frac{1}{s+5}\right)e^{-3s}\right\}$

$y(t) = f(t-3)u(t-3) \rightarrow \quad F(s) = \frac{2}{5}\cdot\frac{1}{s} - \frac{2}{5}\cdot\frac{1}{s+5}$

$$f(t) = \frac{2}{5} - \frac{2}{5}e^{-5t}$$

$$f(t-3) = \frac{2}{5} - \frac{2}{5}e^{-5(t-3)}$$

$$y(t) = \left[\frac{2}{5} - \frac{2}{5}e^{-5(t-3)}\right]u(t-3)$$