7.2 Inverse Transforms

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}\{f(t)\} = F(s) \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$\mathcal{L}^{-1}$ is linear:

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\} = a\mathcal{L}^{-1}\{F(s)\}+b\mathcal{L}^{-1}\{G(s)\}$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{3}{s^2+16} - \frac{1}{s^3}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s^2+4^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+4^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$$

$$= \frac{3}{4}\sin 4t - \frac{1}{2}t^2$$

Ex: Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-5s+12}{s^3+s^2-12s}\right\}$

Partial Fractions

$s^3+s^2-12s = s(s^2+s-12) = s(s+4)(s-3)$

$$\frac{5s^2-5s+12}{s(s+4)(s-3)} = \frac{A}{s}+\frac{B}{s+4}+\frac{C}{s-3}$$

$$\boxed{5s^2-5s+12 = A(s+4)(s-3)+Bs(s-3)+Cs(s+4)}$$

Sub $s=0$: $12 = -12A$

$\boxed{A=-1}$

Sub $s=-4$: $112 = 28B$

$\boxed{B=4}$

Sub $s=3$: $42 = 21C$

$\boxed{C=2}$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s}+\frac{4}{s+4}+\frac{2}{s-3}\right\}$$

Recall $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$

$$= -1+4e^{-4t}+2e^{3t}$$

Cover-Up Method:

Only works for <u>distinct linear factors</u>

$$\frac{5s^2-5s+12}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-3}$$

$$A = \left.\frac{5s^2-5s+12}{(s+4)(s-3)}\right|_{s=0} = \frac{12}{-12} = -1$$

$$B = \left.\frac{5s^2-5s+12}{s(s-3)}\right|_{s=-4} = \frac{112}{28} = 4$$

etc.

Ex: Find $\mathcal{L}^{-1}\left\{\frac{12s^2+6}{s^4+s^2}\right\}$

$$s^4+s^2 = s^2(s^2+1)$$

$$\frac{12s^2+6}{s^2(s^2+1)} = \underbrace{\frac{A}{s}+\frac{B}{s^2}}_{\text{repeated factor}} + \underbrace{\frac{Cs+D}{s^2+1}}_{\text{quadratic factor}}$$

$$12s^2+6 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

Can sub any values and/or match coefficients

Sub $s=0$: $\boxed{6=B}$

Sub $s=i$ (because $i^2+1=0$) Recall $i^2=-1$

$$12(-1)+6 = (Ci+D)(-1)$$

$$-6 = -Ci-D$$

$$-6+0i = -D-Ci$$

$$\boxed{\begin{matrix} D=6 \\ C=0 \end{matrix}}$$

Match s^3 terms:

$$0s^3 = As^3 + Cs^3$$

$$0 = A + C$$

$$\boxed{A = 0}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{s^2} + \frac{6}{s^2+1}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 6t + 6\sin t$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Ex: Use the Laplace Transform to solve

$$\frac{dy}{dt} + 7y = 65\cos 4t \qquad y(0) = 9$$

1) Apply $\mathcal{L}$:

$$\mathcal{L}\{y'(t)\} + \mathcal{L}\{7y\} = \mathcal{L}\{65\cos 4t\}$$

$$sY(s) - y(0) + 7Y(s) = \frac{65s}{s^2+16}$$

$y(0) = 9$: $\quad sY(s) - 9 + 7Y(s) = \frac{65s}{s^2+16}$

2) Solve for $Y(s)$:

$$sY(s) + 7Y(s) = 9 + \frac{65s}{s^2+16}$$

$$(s+7)Y(s) = 9 + \frac{65s}{s^2+16}$$

$$Y(s) = \frac{9}{s+7} + \frac{65s}{(s+7)(s^2+16)}$$

3) Apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{9}{s+7}\right\} + \mathcal{L}^{-1}\left\{\frac{65s}{(s+7)(s^2+16)}\right\}$$

$$y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{65s}{(s+7)(s^2+16)}\right\}$$ ④

Partial Fractions:

$$\frac{65s}{(s+7)(s^2+16)} = \frac{A}{s+7} + \frac{Bs+C}{s^2+16}$$

$$\boxed{65s = A(s^2+16) + (Bs+C)(s+7)}$$

Sub $s=-7$: $-455 = A(65)$

$$\boxed{A=-7}$$

Match s^2 terms:

$$0s^2 = As^2 + Bs^2$$

$$0 = A+B$$

$$\boxed{B=7}$$

Match constants:

$$0 = 16A + 7C$$

$$0 = -112 + 7C$$

$$\boxed{C=16}$$

From ⋆: $y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s+16}{s^2+16}\right\}$

$$y(t) = 9e^{-7t} + \mathcal{L}^{-1}\left\{\frac{-7}{s+7} + \frac{7s}{s^2+4^2} + 4\cdot\frac{4}{s^2+4^2}\right\}$$

$$y(t) = 9e^{-7t} - 7e^{-7t} + 7\cos 4t + 4\sin 4t$$

$$y(t) = 2e^{-7t} + 7\cos 4t + 4\sin 4t$$

Ex: Use the Laplace transform to solve

$y'' - 5y' + 6y = 4e^t \quad y(0)=3, y'(0)=8$

1) Apply $\mathcal{L}$:

$$[s^2Y(s) - sy(0) - y'(0)] - 5[sY(s) - y(0)] + 6Y(s) = \frac{4}{s-1}$$

$y(0)=3$, $y'(0)=8$:

$$s^2Y(s) - 3s - 8 - 5sY(s) + 15 + 6Y(s) = \frac{4}{s-1}$$

2) Solve for $Y(s)$:

$$(s^2-5s+6)Y(s) = 3s-7+\frac{4}{s-1}$$

$$Y(s) = \frac{3s-7}{s^2-5s+6} + \frac{4}{(s+1)(s^2-5s+6)}$$

3) Apply $\mathcal{L}^{-1}$:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3s-7}{(s-2)(s-3)} + \frac{4}{(s-1)(s-2)(s-3)}\right\}$$

Don't do partial fractions twice !!!

$$y(t) = \mathcal{L}^{-1}\left\{\frac{(3s-7)(s-1)+4}{(s-1)(s-2)(s-3)}\right\}$$ ~~(A)~~

$$\frac{3s^2-10s+11}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$3s^2-10s+11 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Sub $s=1$: $4 = 2A$

$\boxed{A=2}$

Sub $s=2$: $3 = -B$

$\boxed{B=-3}$

Sub $s=3$: $8 = 2C$

$\boxed{C=4}$

From ⭐: $y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{3}{s-2} + \frac{4}{s-3}\right\}$

$$y(t) = 2e^{t} - 3e^{2t} + 4e^{3t}$$