7.1 The Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$\mathcal{L}\{f(t)\}$: function of t — $F(s)$: function of s

Notice: F means the transformed version of f

Note: s is restricted to $s > 0$ so that $e^{-st}, te^{-st} \to 0$ as $t \to \infty$

Ex: Find $\mathcal{L}\{t\}$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\,dt$$

	D	I
⊕	t	e^{-st}
⊖	1	$-\frac{1}{s}e^{-st}$
		$\frac{1}{s^2}e^{-st}$

$$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_{t=0}^{t=\infty}$$

(formally, the limit as $t \to \infty$)

$$= 0 - \left[0 - \frac{1}{s^2}(1)\right]$$

$$= \frac{1}{s^2}$$

Ex: Find $\mathcal{L}\{e^{t+7}\}$

$$= \int_0^\infty e^{-st} e^{t+7}\, dt$$

$$= \int_0^\infty e^t \cdot e^7 \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^7 e^{(1-s)t}\, dt$$

$$= \left. \frac{e^7 e^{(1-s)t}}{1-s} \right|_{t=0}^{t=\infty}$$

$$= 0 - \frac{e^7(1)}{1-s}$$

$$= \frac{e^7}{s-1}$$

Ex: Find $\mathcal{L}\{\sin t\}$

$$= \int_0^\infty e^{-st} \sin t \, dt$$

$= uv - \int v \, du$

D	I
(u) $\sin t$	e^{-st} (dv)
$\cos t$	$-\frac{1}{s} e^{-st}$ (v)
(du)	

$$= -\frac{1}{s} e^{-st} \sin t \Big|_{t=0}^{t=\infty} + \int_0^\infty \frac{1}{s} e^{-st} \cos t \, dt$$

$$= 0 + \int_0^\infty \frac{1}{s} e^{-st} \cos t \, dt$$

$= uv - \int v \, du$

D	I
(u) $\cos t$	$\frac{1}{s} e^{-st}$ (dv)
$-\sin t$	$-\frac{1}{s^2} e^{-st}$ (v)
(du)	

$$= -\frac{1}{s^2} e^{-st} \cos t \Big|_{t=0}^{t=\infty} - \int_0^\infty \frac{1}{s^2} e^{-st} \sin t \, dt$$

$$= 0 + \frac{1}{s^2} - \frac{1}{s^2} \int_0^\infty e^{-st} \sin t \, dt$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2} - \frac{1}{s^2} \mathcal{L}\{\sin t\}$$

$$\mathcal{L}\{\sin t\} + \frac{1}{s^2} \mathcal{L}\{\sin t\} = \frac{1}{s^2}$$

$$\left(1+\frac{1}{s^2}\right)\mathcal{L}\{\sin t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{\sin t\} = \frac{\frac{1}{s^2}}{1+\frac{1}{s^2}} \cdot \frac{s^2}{s^2}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

We'll use formula sheet from now on.

$$\mathcal{L}\{t\} = \mathcal{L}\{t^n\}_{n=1} = \frac{1!}{s^2} = \frac{1}{s^2} \checkmark$$

$$\mathcal{L}\{\sin t\} = \mathcal{L}\{\sin \omega t\}_{\omega=1} = \frac{1}{s^2+1} \checkmark$$

$$\boxed{\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$$

Ex: $\mathcal{L}\{7t^2 - 3\cos 2t\} = 7\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\}$

$$= 7 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2+4}$$

$$= \frac{14}{s^3} - \frac{3s}{s^2+4}$$