Series Solutions

Section 6.1 in 9th edition
Section 6.2 in 10th edition

We'll solve $f(x)y'' + g(x)y' + h(x)y = 0$

using $y = \sum_{n=0}^{\infty} c_n x^n$

Goal: Find the coefficients $c_0, c_1, c_2, \ldots$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$ Note: $n \geq 1$
Omit $0c_0x^{-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$ Note: $n \geq 2$
Omit $0c_0x^{-2}$, $0c_1x^{-1}$

Negative exponents will cause problems.
We can omit these terms because they are 0.

Ex: Solve $y'' - xy = 0$ using sigma notation.
Find the first 3 nonzero terms of
y_1 and y_2.

$$\begin{cases} y = \sum_{n=0}^{\infty} c_n x^n \\ y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \end{cases}$$

$$y'' - xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

exponent $k = n-2$
$k \geq 0$

exponent $k = n+1$
$k \geq 1$

Re-index using k.
All series start at the biggest k-value ($k=1$)

$k=0$ term $+ \sum_{k=1}^{\infty}$ $- \sum_{k=1}^{\infty}$ $= 0$

1st term $n=2$ $\quad$ $k=n-2$ $n=k+2$ $\quad$ $k=n+1$ $n=k-1$

$$2C_2 x^0 + \sum_{k=1}^{\infty}(k+2)(k+1)C_{k+2}x^k - \sum_{K=1}^{\infty} C_{k-1}x^k = 0$$

Combine into one series:

$$2C_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)C_{k+2}x^k - C_{k-1}x^k\right] = 0$$

$$2C_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)C_{k+2} - C_{k-1}\right]x^k = 0$$

If $b_0 + b_1 x + b_2 x^2 + \ldots = 0$ for all x
then all coefficients are 0.

Set coefficients $= 0$

$2C_2 = 0$ $\qquad$ $(k+2)(k+1)C_{k+2} - C_{k-1} = 0$

$$C_{k+2} = \frac{C_{k-1}}{(k+2)(k+1)}$$

"recurrence relation"

$$C_{k+2} = \frac{C_{k-1}}{(k+2)(k+1)}, \quad k \geq 1$$

$$C_2 = 0$$

Generate coefficients:

C_0 unknown

C_1 unknown

$C_2 = 0$

$C_3 = \frac{C_0}{3 \cdot 2} = \frac{C_0}{6}$

$C_4 = \frac{C_1}{4 \cdot 3} = \frac{C_1}{12}$

in terms of C_0 and C_1

$C_5 = \frac{C_2}{5 \cdot 4} = 0$

$C_6 = \frac{C_3}{6 \cdot 5} = \frac{1}{30} \cdot \frac{C_0}{6} = \frac{C_0}{180}$

$C_7 = \frac{C_4}{7 \cdot 6} = \frac{1}{42} \cdot \frac{C_1}{12} = \frac{C_1}{504}$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$y = C_0 + C_1 x + \frac{C_0}{6} x^3 + \frac{C_1}{12} x^4 + \frac{C_0}{180} x^6 + \frac{C_1}{504} x^7 + \dots$$

$$y = C_0 \left[1 + \frac{x^3}{6} + \frac{x^6}{180} + \dots \right] \quad (y_1)$$

$$+ C_1 \left[x + \frac{x^4}{12} + \frac{x^7}{504} + \dots \right] \quad (y_2)$$

Notice: We get 2 solutions, y_1 and y_2, because it's a 2nd order DE.

Ex: Solve $y'' - y = 0$, $y(0) = 3$, $y'(0) = 5$ using sigma notation. Find the first six nonzero terms.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$y'' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

exponent $k = n-2$, $k \geq 0$

exponent $k = n$, $k \geq 0$

Re-index using k.
All series start at biggest k-value ($k = 0$)

$$\sum_{k=0}^{\infty} \quad - \sum_{k=0}^{\infty} \quad = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2}x^k - \sum_{k=0}^{\infty} C_k x^k = 0$$

$k = n-2$, $n = k+2$; $k = n$, $n = k$

Combine:

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)C_{k+2}x^k - C_k x^k\right] = 0$$

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1)C_{k+2} - C_k\right] x^k = 0$$

Set coefficients $= 0$

$$(k+2)(k+1)C_{k+2} - C_k = 0$$

$$C_{k+2} = \frac{C_k}{(k+2)(k+1)}, \quad k \geq 0$$

recurrence relation

Use initial Condition:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y = c_0 + c_1 x + c_2 x^2 + \cdots$$

$y=3$, $x=0$: $\boxed{3 = c_0}$

$$y' = c_1 + 2c_2 x + \cdots$$

$y'=5$, $x=0$: $\boxed{5 = c_1}$

Generate Coefficients using recurrence relation:

$$c_2 = \frac{c_0}{2 \cdot 1} = \frac{3}{2}$$

$$c_3 = \frac{c_1}{3 \cdot 2} = \frac{5}{6}$$

$$c_4 = \frac{c_2}{4 \cdot 3} = \frac{1}{12} \cdot \frac{3}{2} = \frac{1}{8}$$

$$c_5 = \frac{c_3}{5 \cdot 4} = \frac{1}{20} \cdot \frac{5}{6} = \frac{1}{24}$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y = 3 + 5x + \frac{3}{2}x^2 + \frac{5}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{24}x^5 + \cdots$$

Ex: Find first 3 nonzero terms of y_1 and y_2 using sigma notation.

$$y'' - (1+x)y = 0$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$y'' - (1+x)y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - (1+x) \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=0}^{\infty} C_n x^n - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

exponent $k = n-2$, $k \geq 0$

exponent $k = n$, $k \geq 0$

exponent $k = n+1$, $k \geq 1$

Re-index using k.
All series start at biggest k-value (k=1)

(k=0 term) $+ \sum_{k=1}^{\infty}$ $-$ (k=0 term) $- \sum_{k=1}^{\infty}$ $- \sum_{k=1}^{\infty}$ $= 0$

1st term, $n=2$:
$2C_2 x^0 + \sum_{k=1}^{\infty} (k+2)(k+1)C_{k+2} x^k$ ($k=n-2$, $n=k+2$)

1st term, $n=0$:
$- C_0 x^0 - \sum_{k=1}^{\infty} C_k x^k$ ($k=n$)

$- \sum_{k=1}^{\infty} C_{k-1} x^k = 0$ ($k=n+1$, $n=k-1$)

Combine:

$$2C_2 - C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2}x^k - C_k x^k - C_{k-1}x^k \right] = 0$$

$$2C_2 - C_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)C_{k+2} - C_k - C_{k-1} \right] x^k = 0$$

Set coefficients = 0

$$2C_2 - C_0 = 0 \qquad (k+2)(k+1)C_{k+2} - C_k - C_{k-1} = 0$$

Recurrence relation:

$$C_{k+2} = \frac{C_k + C_{k-1}}{(k+2)(k+1)}, \quad k \geq 1$$

$$C_2 = \frac{C_0}{2}$$

Generate Coefficients:

C_0 unknown

C_1 unknown

$C_2 = \frac{C_0}{2}$

$$C_3 = \frac{C_1 + C_0}{3 \cdot 2} = \frac{C_1}{6} + \frac{C_0}{6}$$

$$C_4 = \frac{C_2 + C_1}{4 \cdot 3} = \frac{C_2}{12} + \frac{C_1}{12} = \frac{C_0}{24} + \frac{C_1}{12}$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y = C_0 + C_1 x + \frac{C_0}{2} x^2 + \left(\frac{C_1}{6} + \frac{C_0}{6}\right) x^3 + \left(\frac{C_0}{24} + \frac{C_1}{12}\right) x^4 + \cdots$$

$$y = C_0 \left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right] + C_1 \left[x + \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right]$$

$\uparrow$ (y_1) $\qquad$ $\uparrow$ (y_2)

Ex: Find the first 2 nonzero terms of y_1 and y_2 for $y'' - (\sin x)y = 0$

using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

and $y = C_0 + C_1 x + C_2 x^2 + \dots$

$$\begin{cases} y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots \\ y' = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^4 + \dots \\ y'' = 2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots \end{cases}$$

Use enough terms to understand pattern.
Can't predict # of terms needed.

$$y'' - (\sin x)y = 0$$

$$2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots$$

$$-\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)\left(C_0 + C_1 x + C_2 x^2 + \dots\right) = 0$$

$$2C_2 + 6C_3x + 12C_4x^2 + 20C_5x^3 + \dots$$

$$+\left(-x + \frac{x^3}{6} - \frac{x^5}{120} + \dots\right)\left(C_0 + C_1x + C_2x^2 + \dots\right) = 0$$

$$2C_2 + 6C_3x + 12C_4x^2 + 20C_5x^3 + \dots$$

$$-C_0x - C_1x^2 - C_2x^3 + \dots$$

$$\frac{C_0x^3}{6} + \dots$$

Add:

$$2C_2 + (6C_3 - C_0)x + (12C_4 - C_1)x^2 + \left(20C_5 - C_2 + \frac{C_0}{6}\right)x^3 + \dots$$

Set coefficients = 0

$2C_2 = 0 \rightarrow C_2 = 0$

$6C_3 - C_0 = 0 \rightarrow C_3 = \frac{C_0}{6}$

$12C_4 - C_1 = 0 \rightarrow C_4 = \frac{C_1}{12}$

$\vdots$

(C_0 and C_1 are unknown)

$$y = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$y = C_0 + C_1 x + \frac{C_0}{6} x^3 + \frac{C_1}{12} x^4 + \cdots$$

$$y = C_0\left[1 + \frac{x^3}{6} + \cdots\right] + C_1\left[x + \frac{x^4}{12} + \cdots\right]$$

y_1 y_2

Sigma notation method is more powerful:

recurrence relation allows us to generate many terms quickly.