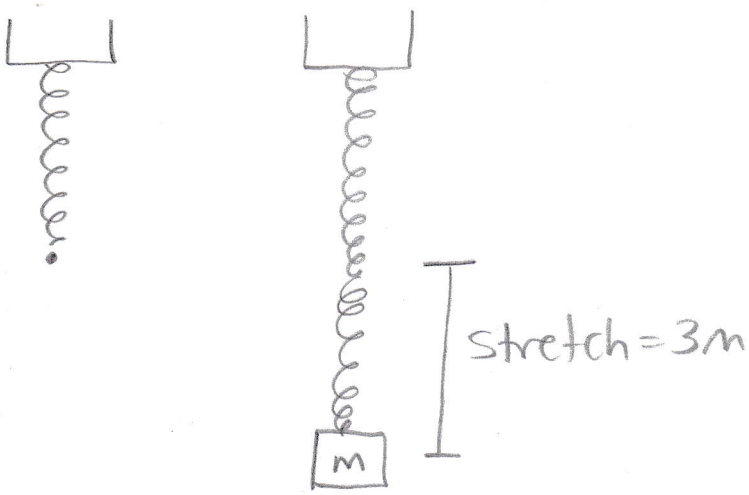
5.1 Spring-Mass Systems

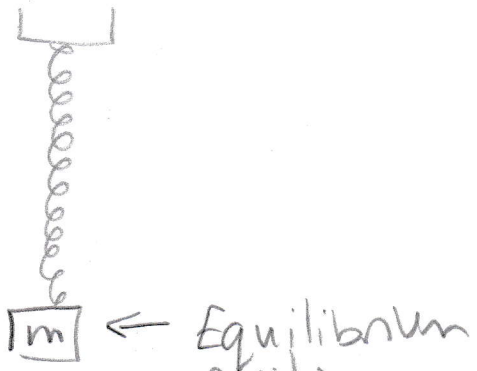


$m = 9\,kg$

$g = 9.8\ N/kg$: $W = 88.2\,N$

$88.2\,N = k(3m)$

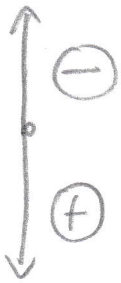
$k = 29.4\ N/m$ "spring constant"



← Equilibrium position

(where mass comes to rest)

x = displacement from equilibrium



Net force $= ma$

$-kx = mx''$

upwards

$$\boxed{mx'' + kx = 0}$$

Ex: A 20kg mass is attached to a spring with spring constant $20/3$ N/m. Mass is released from 1m below equilibrium position with velocity 2m/s upwards.

a) Find $x(t)$

"Periodic Motion" Expect Sin and cos terms.

$$\boxed{mx'' + kx = 0}$$

$$20x'' + 20/3\, x = 0$$

Constant Coefficients (Section 4.3)

$$20m^2 + 20/3 = 0$$

$$20m^2 = -\frac{20}{3}$$

$$m^2 = -\frac{1}{3}$$

$$m = \pm\sqrt{\frac{-1}{3}} \qquad \sqrt{\frac{1}{3}}\sqrt{-1}$$

$$m = \pm\frac{1}{\sqrt{3}}i$$

$$x = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t)$$

Caution: x is a function of t

$$\boxed{x = C_1\cos\frac{1}{\sqrt{3}}t + C_2\sin\frac{1}{\sqrt{3}}t}$$

$x(0)=1$:

$$1 = C_1\cos 0 + C_2\sin 0$$

$$1 = C_1$$

$$\boxed{x = \cos\frac{1}{\sqrt{3}}t + C_2\sin\frac{1}{\sqrt{3}}t}$$

$$x' = -\frac{1}{\sqrt{3}}\sin\frac{1}{\sqrt{3}}t + \frac{1}{\sqrt{3}}C_2\cos\frac{1}{\sqrt{3}}t$$

$x'(0)=-2$:

$$-2 = -\frac{1}{\sqrt{3}}\sin 0 + \frac{1}{\sqrt{3}}C_2\cos 0$$

$$-2 = \frac{1}{\sqrt{3}}C_2$$

$$-2\sqrt{3} = C_2$$

$$\boxed{x = \cos\frac{1}{\sqrt{3}}t - 2\sqrt{3}\sin\frac{1}{\sqrt{3}}t}$$

b) Write $x(t)$ as $A\sin\left(\frac{t}{\sqrt{3}}+\phi\right)$

$\cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}} = A\sin\left(\frac{t}{\sqrt{3}}+\phi\right)$ Find A, ϕ

Formula sheet: $\sin\left(\frac{t}{\sqrt{3}}+\phi\right) = \sin\frac{t}{\sqrt{3}}\cos\phi + \cos\frac{t}{\sqrt{3}}\sin\phi$

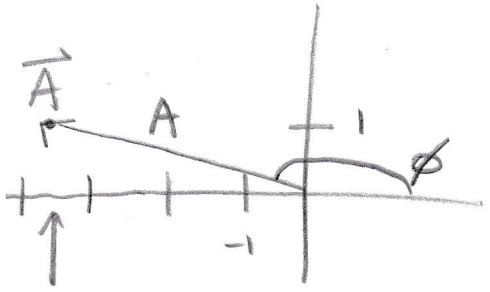
$\cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}} = A\sin\frac{t}{\sqrt{3}}\cos\phi + A\cos\frac{t}{\sqrt{3}}\sin\phi$

$A\cos\phi = -2\sqrt{3}$

$A\sin\phi = 1$

Shortcut: $A\sin\phi$ is the coefficient of cos term
$A\cos\phi$ " " sin term

Draw $\vec{A}$:



$-2\sqrt{3} \approx -3.5$

$A = \sqrt{1^2 + (-2\sqrt{3})^2} = \sqrt{13}$

$\phi = \tan^{-1}\left(\frac{1}{-2\sqrt{3}}\right)$ $(+\pi ?)$

$= \tan^{-1}\left(\frac{1}{-2\sqrt{3}}\right) + \pi$

≈ 2.86 rads

$$x(t) = A\sin\left(\frac{t}{\sqrt{3}} + \phi\right)$$

$$\boxed{x(t) = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)}$$

c) Mass's maximum displacement from equilibrium position?

$$\boxed{\pm\sqrt{13}\text{ m}}$$

because $\sqrt{13}\sin(\sim)$ ranges from $-\sqrt{13}$ to $\sqrt{13}$

d) Period of motion?

$$x = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$

$$\text{Period} = \frac{2\pi}{\left(\frac{1}{\sqrt{3}}\right)} = 2\pi\sqrt{3}\text{ seconds/cycle}$$

e) # of complete cycles after 60s?

$$\frac{60\text{ s}}{2\pi\sqrt{3}\text{ s/cycle}} \approx 5.5\text{ cycles}$$

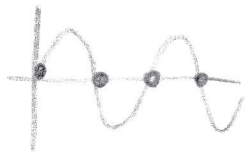
$$\boxed{5\text{ complete cycles}}$$

f) time when mass first returns to equilibrium position?

$$x = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$

$x = 0$:

$$0 = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$



$\frac{t}{\sqrt{3}} + 2.86 = 0, \pi, 2\pi, \ldots$

$\frac{t}{\sqrt{3}} + 2.86 = 0$ nonsense

$\frac{t}{\sqrt{3}} + 2.86 = \pi$

$t \approx 0.49 \text{ Å}$

Damped Motion

$$mx'' + \beta x' + kx = 0$$

β: resistance is proportional to velocity

Ex: $m = 20\text{kg}$ $k = 20/3$ N/m

Damping force of magnitude 25 times velocity

Mass is initially at equilibrium, heading downwards at 4m/s. Find $x(t)$.

$$mx'' + \beta x' + kx = 0$$

$$20x'' + 25x' + \frac{20}{3}x = 0$$

$$20m^2 + 25m + \frac{20}{3} = 0$$

$$60m^2 + 75m + 20 = 0$$

$$\boxed{12m^2 + 15m + 4 = 0}$$

$$m = \frac{-15 \pm \sqrt{15^2 - 4 \cdot 12 \cdot 4}}{24}$$

$$m = \frac{-15 \pm \sqrt{33}}{24}$$

$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

Caution: x is a function of t

$$\boxed{x = C_1 e^{\frac{-15+\sqrt{33}}{24} t} + C_2 e^{\frac{-15-\sqrt{33}}{24} t}}$$

$x(0)=0$: $\quad 0 = C_1 + C_2 \quad (1)$

$$x' = \frac{-15+\sqrt{33}}{24} C_1 e^{\frac{-15+\sqrt{33}}{24} t} + \frac{-15-\sqrt{33}}{24} C_2 e^{\frac{-15-\sqrt{33}}{24} t}$$

$x'(0)=4$:

$$4 = \frac{-15+\sqrt{33}}{24} C_1 + \frac{-15-\sqrt{33}}{24} C_2 \quad (2)$$

(1): $C_1 = -C_2 \rightarrow$ (2):

$$4 = \frac{-15+\sqrt{33}}{24} (-C_2) + \frac{-15-\sqrt{33}}{24} C_2$$

$$4 = -\frac{2\sqrt{33}}{24} C_2$$

$$C_2 = -\frac{48}{\sqrt{33}}$$

$$C_1 = \frac{48}{\sqrt{33}}$$

$$\boxed{x = \frac{48}{\sqrt{33}} e^{\frac{-15+\sqrt{33}}{24} t} - \frac{48}{\sqrt{33}} e^{\frac{-15-\sqrt{33}}{24} t}}$$

$$F = m \cdot a$$

Metric: $N = kg \cdot m/s^2$

Imperial: $lb = slug \cdot ft/s^2$ $\quad g = 32\ ft/s^2$

Ex: Weight of 3.2 lbs stretches a spring 1.6 ft. Damping force of magnitude 0.4 times velocity.

$x(0) = -1 \quad x'(0) = 0$

a) Find $x(t)$

$$mx'' + \beta x' + kx = 0$$

m: $\frac{3.2\ lbs}{32\ ft/s^2} = 0.1\ slugs$

β: 0.4

k: $3.2\ lbs = k(1.6\ ft)$, $k = 2\ lbs/ft$

$$0.1x'' + 0.4x' + 2x = 0$$

$$0.1m^2 + 0.4m + 2 = 0$$

$$m^2 + 4m + 20 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 20}}{2}$$

$$m = \frac{-4 \pm \sqrt{-64}}{2} \qquad \sqrt{64}\sqrt{-1}$$

$$m = \frac{-4 \pm 8i}{2}$$

$$m = -2 \pm 4i$$

$$x = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

Caution: x is a function of t

$$x = e^{-2t}(C_1 \cos 4t + C_2 \sin 4t)$$

$x(0) = -1$: $\quad -1 = C_1$

$$x = e^{-2t}(-\cos 4t + C_2 \sin 4t)$$

$$x' = e^{-2t}(4 \sin 4t + 4C_2 \cos 4t) - 2e^{-2t}(-\cos 4t + C_2 \sin 4t)$$

$x'(0) = 0$: $\quad 0 = 4C_2 - 2(-1)$

$$C_2 = -\frac{1}{2}$$

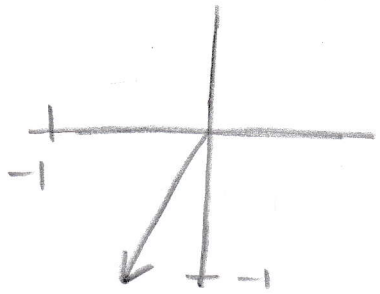
$$x = e^{-2t}\left(-\cos 4t - \frac{1}{2}\sin 4t\right)$$

b) When does the mass first pass through the equilibrium position, heading upwards?

Rewrite $-\cos 4t - \frac{1}{2}\sin 4t = A\sin(4t+\phi)$

$A\sin\phi = -1$

$A\cos\phi = -\frac{1}{2}$



$$A = \sqrt{(-1)^2 + \left(\frac{-1}{2}\right)^2}$$
$$= \sqrt{\frac{5}{4}}$$
$$= \frac{\sqrt{5}}{2}$$

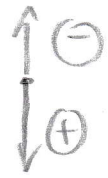
$$\phi = \tan^{-1}\left(\frac{-1}{\left(\frac{-1}{2}\right)}\right) \quad (+\pi\ ?)$$

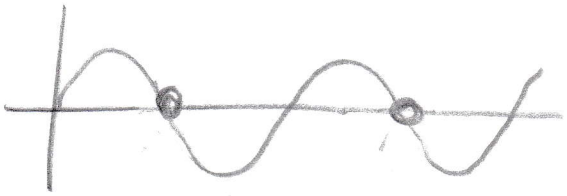
$$= \tan^{-1}(2) + \pi$$
$$\approx 4.25 \text{ rads}$$

$$x = e^{-2t} A\sin(4t+\phi)$$

$$\boxed{x = \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.25)}$$

Want $x=0$, becoming negative





$4t + 4.25 = \pi, 3\pi, \ldots$

$4t + 4.25 = \pi$ transverse

$4t + 4.25 = 3\pi$

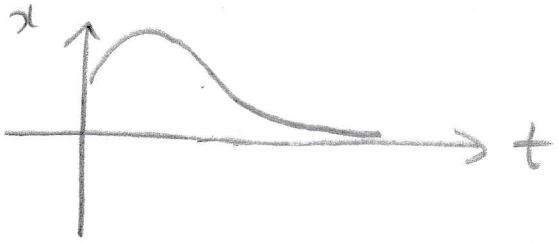
$t \approx 1.29$ s

Consider damped motion

$$mx'' + \beta x' + kx = 0$$

3 Scenarios:

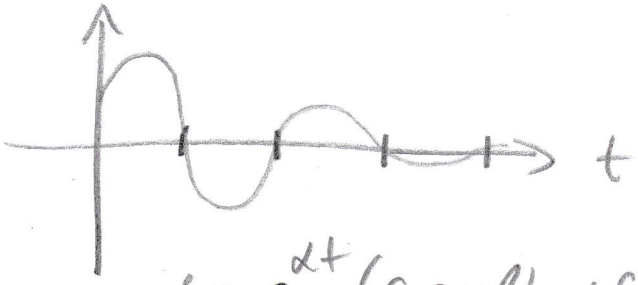
1) Overdamped motion



$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

2 distinct roots m_1, m_2

2) Underdamped motion



$$x = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

Complex roots $\alpha \pm \beta i$

3) Critically damped motion
Overdamped, but borderline

$$x = e^{m_1 t}(C_1 + C_2 t)$$

1 repeated root

Driven Motion

$$mx'' + \beta x' + kx = f(t)$$

$f(t)$ ← external force

Brief Ex: $x'' + 2x' + 5x = \cos t$

$x(0) = 0 \quad x'(0) = 0$

1) x_c? Similar to Section 4.4 but x is a function of t

$x'' + 2x' + 5x = 0$

$m^2 + 2m + 5 = 0$

$m = -1 \pm 2i$

$x_c = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

$x_c = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$

2) $x_p = A\cos t + B\sin t$ | not bad case

3) $x_p \to DE$

$x_p = \frac{1}{5}\cos t + \frac{1}{10}\sin t$

4) $x = x_c + x_p$

$x = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + \frac{1}{5}\cos t + \frac{1}{10}\sin t$

5) Use initial conditions

$$x = \underbrace{e^{-t}\left(-\tfrac{1}{5}\cos 2t - \tfrac{3}{10}\sin 2t\right)}_{\to 0 \text{ as } t \to \infty} + \frac{1}{5}\cos t + \frac{1}{10}\sin t$$