4.7 Cauchy-Euler DE's

Ex: Solve $x^2y'' - 7xy' + 15y = 0$

(Power of x matches order of derivative)

"Cauchy-Euler DE"

auxiliary equation:

$x^3y''' \rightarrow m(m-1)(m-2)$

$x^2y'' \rightarrow m(m-1)$

$xy' \rightarrow m$

$y \rightarrow 1$

$$m(m-1) - 7m + 15 = 0$$

$$m^2 - 8m + 15 = 0$$

$$(m-3)(m-5) = 0$$

$$m = 3, 5$$

Roots	Solutions
distinct real # $m_1, m_2, m_3, \cdots$	$x^{m_1}, x^{m_2}, x^{m_3}, \cdots$
repeated real # $m_1, m_1, m_1, \cdots$	$x^{m_1}, x^{m_1}\ln x, x^{m_1}(\ln x)^2, \cdots$
complex # $\alpha \pm \beta i$	$x^{\alpha}[C_1\cos(\beta \ln x) + C_2\sin(\beta \ln x)]$

$y_1 = x^3 \quad y_2 = x^5$

General Solution $\boxed{y = C_1 x^3 + C_2 x^5}$

Check:

$$\begin{cases} y = C_1 x^3 + C_2 x^5 \\ y' = 3C_1 x^2 + 5C_2 x^4 \\ y'' = 6C_1 x + 20C_2 x^3 \end{cases}$$

$$LS = x^2 y'' - 7x y' + 15y$$

$$= x^2 (6C_1 x + 20C_2 x^3) - 7x (3C_1 x^2 + 5C_2 x^4) + 15 (C_1 x^3 + C_2 x^5)$$

$$= (6 - 21 + 15) C_1 x^3 + (20 - 35 + 15) C_2 x^5$$

$$= 0 C_1 x^3 + 0 C_2 x^5$$

$$= 0$$

$$= RS \checkmark$$

Coefficient is 0 because 3 is a root of the auxiliary equation

	Constant Coefficient DE's (Section 4.3)	Cauchy-Euler DE's
distinct roots $m_1, m_2, \ldots$	$e^{m_1 x}, e^{m_2 x}, \ldots$	$x^{m_1}, x^{m_2}, \ldots$
repeated roots $m_1, m_1, m_1, \ldots$	$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots$	$x^{m_1}, x^{m_1} \ln x, x^{m_1} (\ln x)^2, \ldots$
complex roots $\alpha \pm \beta i$	$e^{\alpha x}[C_1 \cos \beta x + C_2 \sin \beta x]$	$x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

$$\boxed{x \to \ln x}$$

$$e^{m_1 x} \to e^{m_1 \ln x}$$

$$= e^{\ln x^{m_1}}$$

$$= x^{m_1}$$

Ex: Solve $x^2y'' - 3xy' + 4y = 0$

$y(1) = 6 \quad y'(1) = 3$

Cauchy-Euler $\quad m(m-1) - 3m + 4 = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$y_1 = x^2 \quad y_2 = x^2 \ln x$

$$\boxed{y = C_1 x^2 + C_2 x^2 \ln x}$$

$x = 1$
$y = 6$: $\quad 6 = C_1$

$$\boxed{y = 6x^2 + C_2 x^2 \ln x}$$

$$y' = 12x + C_2\left(x^2 \cdot \frac{1}{x} + 2x \ln x\right)$$

$x = 1$
$y' = 3$: $\quad 3 = 12 + C_2(1)$

$-9 = C_2$

$$\boxed{y = 6x^2 - 9x^2 \ln x}$$

Ex: Solve $x^2y'' - xy' + 4y = 0$

Cauchy-Euler $m(m-1) - m + 4 = 0$

$$m^2 - 2m + 4 = 0$$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2} \quad \sqrt{4}\sqrt{3}\sqrt{-1}$$

$$m = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$m = 1 \pm \sqrt{3}\,i$$

$$y = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$y = x\left[C_1 \cos(\sqrt{3} \ln x) + C_2 \sin(\sqrt{3} \ln x)\right]$$

Ex: Solve $x^3y''' + 3x^2y'' + 3xy' - 3y = 0$

Cauchy-Euler $m(m-1)(m-2) + 3m(m-1) + 3m - 3 = 0$

$m(m^2-3m+2) + 3(m^2-m) + 3m - 3 = 0$

$m^3 - 3m^2 + 2m + 3m^2 - 3m + 3m - 3 = 0$

$$\boxed{m^3 + 2m - 3 = 0}$$

Rational Roots: $p: \pm 1, \pm 3$

$q: \pm 1$

$\frac{p}{q}: \pm 1, \pm 3$

Possible roots

Test $m = -1$: $(-1)^3 + 2(-1) - 3 = 0$?

No

$m = 1$: $1^3 + 2(1) - 3 = 0$?

yes

$(m-1)(\ ?\ ) = 0$

$m = 1$ is a root

$(m-1)(m^2 + ?\, m + 3) = 0$

$(m-1)(m^2 + m + 3) = 0$

$m = 1$ $\qquad m^2 + m + 3 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 3}}{2}$$

$$m = \frac{-1 \pm \sqrt{-11}}{2} \quad \sqrt{11}\sqrt{-1}$$

$$m = \frac{-1 \pm \sqrt{11}\, i}{2}$$

$$m = \underbrace{\frac{-1}{2} \pm \frac{\sqrt{11}}{2} i}$$

$$x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$\boxed{y = x^{-1/2}\left[C_1 \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + C_2 \sin\left(\frac{\sqrt{11}}{2} \ln x\right)\right] + C_3 x}$$

Ex: Solve $x^2y'' - 6xy' + 12y = \frac{1}{x}$

~~Note: Can use Undetermined Coefficients (4.4) for Variation of Parameters (4.6)~~

To solve non-homogeneous Cauchy-Euler DE's we'll use Variation of Parameters.

1) y_c? $\quad x^2y'' - 6xy' + 12y = 0$

Cauchy-Euler $\quad m(m-1) - 6m + 12 = 0$

$m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0$

$m = 3, 4$

$y_1 = x^3 \quad y_2 = x^4$

$y_c = C_1x^3 + C_2x^4$

2) $W = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix} = x^6$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$

$= \begin{vmatrix} 0 & x^4 \\ x^{-3} & 4x^3 \end{vmatrix}$

$= -x$

Caution: Standard Form

$y'' - \frac{6}{x}y' + \frac{12}{x^2}y = \frac{1}{x^3}$

$f(x) = x^{-3}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Refer to standard form for $f(x)$

$$= \begin{vmatrix} x^3 & 0 \\ 3x^2 & x^{-3} \end{vmatrix}$$

$$= 1$$

3) $u_1' = \frac{W_1}{W} = -x^{-5}$

$$u_1 = \int -x^{-5}\,dx$$

$$= \frac{x^{-4}}{4}$$

4) $u_2' = \frac{W_2}{W} = x^{-6}$

$$u_2 = \int x^{-6}\,dx$$

$$= \frac{-x^{-5}}{5}$$

5) $y_p = u_1 y_1 + u_2 y_2$

$$y_p = \frac{x^{-4}}{4} \cdot x^3 - \frac{x^{-5}}{5} \cdot x^4$$

$$y_p = \frac{1}{4} \cdot \frac{1}{x} - \frac{1}{5} \cdot \frac{1}{x}$$

$$y_p = \frac{1}{20} \cdot \frac{1}{x}$$

6) General Solution

$$y = y_c + y_p$$

$$y = C_1 x^3 + C_2 x^4 + \frac{1}{20x}$$