# 4.6 Variation of Parameters

Find a particular solution $y_p$ for

$y'' + ay' + by = f(x)$ when

Section 4.4 doesn't apply.

Idea: Find solutions $y_1, y_2$ to $y'' + ay' + by = 0$

let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

Integrate to get $u_1$ and $u_2$

Then $y_p = u_1 y_1 + u_2 y_2$

Ex: Solve $5y'' + 20y = 5\csc 2x$

$$\boxed{y'' + 4y = \csc 2x}$$ Standard Form

1) $y_c$? $\quad y'' + 4y = 0$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$m = \pm 2i$

$y_c = C_1 \cos 2x + C_2 \sin 2x$

$y_1 = \cos 2x \quad y_2 = \sin 2x$

2) $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ Caution: Use standard form $f(x) = \csc 2x$

$= \begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2\cos 2x \end{vmatrix}$

$= -\sin 2x \cdot \csc 2x$

$= -1$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$ Caution: Use standard form $f(x) = \csc 2x$

$$= \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \csc 2x \end{vmatrix}$$

$$= \cos 2x \cdot \csc 2x$$

$$= \cot 2x$$

3) $$u_1' = \frac{W_1}{W}$$

$$= \frac{-1}{2}$$

$$u_1 = \int \frac{-1}{2}\, dx$$

$$u_1 = \frac{-1}{2} x$$

Don't use a constant here

4) $$u_2' = \frac{W_2}{W}$$

$$= \frac{\cot 2x}{2}$$

$$u_2 = \int \frac{\cot 2x}{2}\, dx$$

$$= \frac{1}{4} \ln |\sin 2x|$$

5) $y_p = u_1 y_1 + u_2 y_2$

$= -\frac{1}{2} x \cos 2x + \frac{1}{4}(\ln|\sin 2x|)\sin 2x$

6) General Solution

$y = y_c + y_p$

$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \ln|\sin 2x|$

Ex: Solve $3y'' + 15y' + 18y = 3\cos e^x$

$\boxed{y'' + 5y' + 6y = \cos e^x}$ standard form

$f(x)$

1) $y_c$? $\quad y'' + 5y' + 6y = 0$

$m^2 + 5m + 6 = 0$

$(m+2)(m+3) = 0$

$m = -2, -3$

$y_c = C_1 e^{-2x} + C_2 e^{-3x}$

$y_1 = e^{-2x} \quad y_2 = e^{-3x}$

2) $W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$

$= -3e^{-5x} + 2e^{-5x}$

$= -e^{-5x}$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ Refer to standard form to get $f(x)$

$= \begin{vmatrix} 0 & e^{-3x} \\ \cos e^x & -3e^{-3x} \end{vmatrix}$

$= -e^{-3x} \cos e^x$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

$= \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \cos e^x \end{vmatrix}$

$= e^{-2x} \cos e^x$

3) $u_1' = \frac{W_1}{W}$

$= \frac{-e^{-3x}\cos e^x}{-e^{-3x}}$

$= e^{2x}\cos e^x$

$u_1 = \int e^{2x}\cos e^x\, dx$

$= \int e^x \cdot \cos e^x \cdot e^x\, dx$

$= \int u\cos u\, du$

$= u\sin u + \cos u$

$= e^x \sin e^x + \cos e^x$

$u = e^x$
$du = e^x dx$

| | D | I |
|---|---|---|
| + | $u$ | $\cos u$ |
| − | $1$ | $\sin u$ |
| | | $-\cos u$ |

4) $u_2' = \frac{W_2}{W}$

$= \frac{e^{-2x}\cos e^x}{-e^{-3x}}$

$= -e^{3x}\cos e^x$

$$u_2 = \int -e^{3x}\cos e^x \, dx$$

$$= \int -e^{2x}\cos e^x \cdot e^x \, dx$$

$$= \int -u^2 \cos u \, du$$

$u = e^x$
$du = e^x dx$

| | D | I |
|---|---|---|
| ⊕ | $-u^2$ | $\cos u$ |
| ⊖ | $-2u$ | $\sin u$ |
| ⊕ | $-2$ | $-\cos u$ |
| | | $-\sin u$ |

$$= -u^2\sin u - 2u\cos u + 2\sin u$$

$$= -e^{2x}\sin e^x - 2e^x\cos e^x + 2\sin e^x$$

5) $y_p = u_1 y_1 + u_2 y_2$

$$= \left(e^x \sin e^x + \cos e^x\right)e^{-2x} + \left(-e^{2x}\sin e^x - 2e^x\cos e^x + 2\sin e^x\right)e^{-3x}$$

$$= e^{-x}\sin e^x + e^{-2x}\cos e^x - e^{-x}\sin e^x - 2e^{-2x}\cos e^x + 2e^{-3x}\sin e^x$$

$$= -e^{-2x}\cos e^x + 2e^{-3x}\sin e^x$$

6) $y = y_c + y_p$

$$y = C_1 e^{-2x} + C_2 e^{-3x} - e^{-2x}\cos e^x + 2e^{-3x}\sin e^x$$

Ex: Solve $y'' - 3y' + 2y = x^3$

Standard Form ✓

Could use Section 4.4 with $y_p = Ax^3 + Bx^2 + Cx + D$ but Variation of Parameters may be faster.

1) $y_c$? $y'' - 3y' + 2y = 0$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$y_c = C_1 e^x + C_2 e^{2x}$

$y_1 = e^x \quad y_2 = e^{2x}$

2) $W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$

$= 2e^{3x} - e^{3x}$

$= e^{3x}$

$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$

$= \begin{vmatrix} 0 & e^{2x} \\ x^3 & 2e^{2x} \end{vmatrix}$

$= -x^3 e^{2x}$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & x^3 \end{vmatrix}$$

$$= x^3 e^x$$

3) $u_1' = \dfrac{W_1}{W}$

$$= \frac{-x^3 e^{2x}}{e^{3x}}$$

$$= -x^3 e^{-x}$$

| | D | I |
|---|---|---|
| ⊕ | $-x^3$ | $e^{-x}$ |
| ⊖ | $-3x^2$ | $-e^{-x}$ |
| ⊕ | $-6x$ | $e^{-x}$ |
| ⊖ | $-6$ | $-e^{-x}$ |
| | | $e^{-x}$ |

$$u_1 = (x^3 + 3x^2 + 6x + 6)e^{-x}$$

4) $u_2' = \dfrac{W_2}{W}$

$= \dfrac{x^3 e^x}{e^{3x}}$

$= x^3 e^{-2x}$

| | D | I |
|---|---|---|
| ⊕ | $x^3$ | $e^{-2x}$ |
| ⊖ | $3x^2$ | $-\frac{1}{2}e^{-2x}$ |
| ⊕ | $6x$ | $\frac{1}{4}e^{-2x}$ |
| ⊖ | $6$ | $-\frac{1}{8}e^{-2x}$ |
| | | $\frac{1}{16}e^{-2x}$ |

$u_2 = \left(\frac{-x^3}{2} - \frac{3x^2}{4} - \frac{6x}{8} - \frac{6}{16}\right)e^{-2x}$

5) $y_p = u_1 y_1 + u_2 y_2$

$= \left(x^3+3x^2+6x+6\right)e^{-x}\cdot e^{x}$

$+\left(\frac{-x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}\right)e^{-2x}\cdot e^{2x}$

$= x^3+3x^2+6x+6 - \frac{x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}$

$= \frac{x^3}{2} + \frac{9x^2}{4} + \frac{21x}{4} + \frac{45}{8}$

6) $y = y_c + y_p \therefore y = C_1 e^{x} + C_2 e^{2x} + \frac{x^3}{2} + \frac{9x^2}{4} + \frac{21x}{4} + \frac{45}{8}$