4.4 Nonhomogeneous Linear DE's with Constant Coefficients

Ex: Solve $y'' - 5y' + 6y = 5x$

1) Solve $y'' - 5y' + 6y = 0$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$y_c = C_1 e^{2x} + C_2 e^{3x}$$

"complementary solution"

2) Find y_p, the particular solution

$$f(x) = 5x$$

y_p consists of all terms in $f(x), f'(x), f''(x), \ldots$

$$f(x) = 5x$$
$$f'(x) = 5$$
$$f''(x) = 0$$

$$\boxed{y_p = Ax + B}$$

Note: y_p has unknown coefficients

$f(x)$	y_p
polynomial of degree n	$A_n x^n + \dots A_1 x + A_0$
$\#\cos kx + \#\sin kx$	$A\cos kx + B\sin kx$
$\#e^{kx}$	Ae^{kx}

3) Put $y_p \to DE$ (Find A and B)

$$\begin{cases} y_p = Ax + B \\ y_p' = A \\ y_p'' = 0 \end{cases}$$

$$y'' - 5y' + 6y = 5x$$

$$-5A + 6(Ax + B) = 5x$$

$$6Ax + 6B - 5A = 5x + 0$$

Match coefficients

$$6A = 5 \quad (1)$$

$$6B - 5A = 0 \quad (2)$$

(1): $\boxed{A = \frac{5}{6}}$

$A = \frac{5}{6} \to (2)$: $6B - 5\left(\frac{5}{6}\right) = 0$

$6B = \frac{25}{6}$ $\boxed{B = \frac{25}{36}}$

$y_p = Ax + B$

$$\boxed{y_p = \frac{5}{6}x + \frac{25}{36}}$$

4) General Solution

$y = y_c + y_p$

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x} + \frac{5}{6}x + \frac{25}{36}}$$

5) Use ~~any~~ initial conditions

Ex: Solve $y'' - 4y' + 4y = 2\sin 3x$

1) $y'' - 4y' + 4y = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$y_c = C_1 e^{2x} + C_2 x e^{2x}$

2) Find y_p

$f(x) = 2\sin 3x$

$f'(x) = 6\cos 3x$

$f''(x) = -18\sin 3x$

$\vdots$

$$\boxed{y_p = A\cos 3x + B\sin 3x}$$

3) $y_p \rightarrow DE$ (Find A and B)

$$\begin{cases} y_p = A\cos 3x + B\sin 3x \\ y_p' = -3A\sin 3x + 3B\cos 3x \\ y_p'' = -9A\cos 3x - 9B\sin 3x \end{cases}$$

$$y'' - 4y' + 4y = 2\sin 3x$$

$$(-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos 3x) + 4(A\cos 3x + B\sin 3x) = 2\sin 3x$$

$$(-9A - 12B + 4A)\cos 3x + (-9B + 12A + 4B)\sin 3x = 2\sin 3x$$

$$(-5A - 12B)\cos 3x + (12A - 5B)\sin 3x = 0\cos 3x + 2\sin 3x$$

$$-5A - 12B = 0 \quad (1)$$
$$12A - 5B = 2 \quad (2)$$

$12 \times (1)$: $-60A - 144B = 0$

$5 \times (2)$: $60A - 25B = 10$

$$+ \quad -169B = 10$$

$$\boxed{B = \frac{-10}{169}}$$

$B = \frac{-10}{169} \rightarrow$ ①: $\quad -5A - 12\left(\frac{-10}{169}\right) = 0$

$$-5A = \frac{-120}{169}$$

$$\boxed{A = \frac{24}{169}}$$

$$\boxed{y_p = \frac{24}{169}\cos 3x - \frac{-10}{169}\sin 3x}$$

4) $y = y_c + y_p$

$$\boxed{y = C_1 e^{2x} + C_2 x e^{2x} + \frac{24}{169}\cos 3x - \frac{-10}{169}\sin 3x}$$

5) Initial conditions

Bad Case

Any terms in y_p that also appear in y_c must be multiplied by the smallest power of x that eliminates the duplication.

Ex: Given $y_c = C_1 e^x + C_2 x e^x$

Find y_p if:

a) $f(x) = 5e^x + 6e^{2x}$

$f'(x) = 5e^x + 12e^{2x}$

$\vdots$

$y_p = Ae^x + Be^{2x}$ ← no duplication

this term appears in y_c "duplication"

$$\boxed{y_p = Ax^2e^x + Be^{2x}}$$

b) $f(x) = e^x \cos x + 1$

$f'(x) = -e^x \sin x + e^x \cos x$

$\vdots$

$y_p = Ae^x \cos x + Be^x \sin x + C$

no duplication

c) $f(x) = 7xe^x$

$f'(x) = 7xe^x + 7e^x$

$\vdots$

$y_p = Axe^x + Be^x$

both terms are duplicated

Bx^2e^x

Ax^3e^x

$y_p = Ax^3e^x + Bx^2e^x$

Ex: Solve $y'' - 6y' + 8y = 3e^{2x}$

$y(0) = 0 \quad y'(0) = 0$

1) $y'' - 6y' + 8y = 0$

$m^2 - 6m + 8 = 0$

$(m-2)(m-4) = 0$

$m = 2, 4$

$y_c = C_1e^{2x} + C_2e^{4x}$

2) Find y_p

$f(x) = 3e^{2x}$

$f'(x) = 6e^{2x}$

$\vdots$

~~$y_p = Ae^{2x}$~~ duplication

$y_p = Axe^{2x}$

3) $y_p \rightarrow$ DE (Find A)

$\boxed{y_p = (Ax)e^{2x}}$

$\boxed{y_p' = (2Ax)e^{2x} + Ae^{2x}}$

$y_p'' = (4Ax)e^{2x} + 2Ae^{2x} + 2Ae^{2x}$

$\boxed{y_p'' = 4Axe^{2x} + 4Ae^{2x}}$

$y'' - 6y' + 8y = 3e^{2x}$

$(4Axe^{2x} + 4Ae^{2x}) - 6[2Axe^{2x} + Ae^{2x}]$

$+ 8Axe^{2x} = 3e^{2x}$

$(4A - 12A + 8A)xe^{2x} + (4A - 6A)e^{2x} = 3e^{2x}$

$-2Ae^{2x} = 3e^{2x}$

$-2A = 3$

$A = -3/2$

$$\boxed{y_p = -\frac{3}{2} x e^{2x}}$$

4) $y = y_c + y_p$

$$\boxed{y = C_1 e^{2x} + C_2 e^{4x} - \frac{3}{2} x e^{2x}}$$

5) $\begin{matrix} x=0 \\ y=0 \end{matrix}$; $0 = C_1 + C_2$ (1)

$$y' = 2C_1 e^{2x} + 4C_2 e^{4x} - \frac{3}{2}\left[2x e^{2x} + e^{2x}\right]$$

$\begin{matrix} x=0 \\ y'=0 \end{matrix}$: $0 = 2C_1 + 4C_2 - \frac{3}{2}$

$$2C_1 + 4C_2 = \frac{3}{2} \quad (2)$$

$-2 \times (1)$: $\quad -2C_1 - 2C_2 = 0$

(2) : $\quad 2C_1 + 4C_2 = \frac{3}{2}$

$+$

$$2C_2 = 3/2$$

$$\boxed{C_2 = 3/4}$$

$C_2 = \frac{3}{4} \rightarrow (1)$: $\boxed{C_1 = -\frac{3}{4}}$

$$\boxed{y = -\frac{3}{4} e^{2x} + \frac{3}{4} e^{4x} - \frac{3}{2} x e^{2x}}$$

Ex: Solve $y'' - y = xe^{4x}$

1) $y'' - y = 0$

$m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m = 1, -1$

$y_c = C_1 e^{x} + C_2 e^{-x}$

2) Find y_p

$f(x) = xe^{4x}$

$f'(x) = 4xe^{4x} + e^{4x}$

$\vdots$

$y_p = Axe^{4x} + Be^{4x}$ no duplication ☺

3) $y_p \to DE$ (Find A, B)

$$\begin{cases} y_p = (Ax)e^{4x} + Be^{4x} \\ y_p' = (4Ax)e^{4x} + Ae^{4x} + 4Be^{4x} \\ y_p'' = (16Axe^{4x} + 4Ae^{4x}) + 4Ae^{4x} + 16Be^{4x} \end{cases}$$

$\boxed{y_p'' = 16Axe^{4x} + 8Ae^{4x} + 16Be^{4x}}$

$$y'' - y = xe^{4x}$$

$$\left(16Axe^{4x} + 8Ae^{4x} + 16Be^{4x}\right) - \left(Axe^{4x} + Be^{4x}\right) = xe^{4x}$$

$$(15A)xe^{4x} + (8A + 15B)e^{4x} = xe^{4x} + 0e^{4x}$$

$$15A = 1 \quad (1)$$
$$8A + 15B = 0 \quad (2)$$

(1): $A = \frac{1}{15}$

$A = \frac{1}{15} \rightarrow$ (2): $8A + 15B = 0$

$$\frac{8}{15} + 15B = 0$$
$$15B = \frac{-8}{15}$$
$$B = \frac{-8}{225}$$

$$\boxed{y_p = \frac{1}{15}xe^{4x} - \frac{8}{225}e^{4x}}$$

4) $y = y_c + y_p$

$$y = C_1e^{x} + C_2e^{-x} + \frac{1}{15}xe^{4x} - \frac{8}{225}e^{4x}$$

5) Initial conditions