4.3 Homogeneous Linear DE's with Constant Coefficients

Ex: Solve $y'' - 4y' - 21y = 0$

auxiliary equation $m^2 - 4m - 21 = 0$

$(m-7)(m+3) = 0$

$m = 7, -3$

Roots	Solutions
distinct real # $m_1, m_2, m_3, \cdots$	$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \cdots$
repeated real # $m_1, m_1, m_1, \cdots$	$e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \cdots$
Complex # $\alpha \pm \beta i$	$e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$

$y_1 = e^{7x}$ $y_2 = e^{-3x}$

General solution $y = C_1 y_1 + C_2 y_2$

$$\boxed{y = C_1 e^{7x} + C_2 e^{-3x}}$$

Check:
$$\begin{cases} y = C_1 e^{7x} + C_2 e^{-3x} \\ y' = 7C_1 e^{7x} - 3C_2 e^{-3x} \\ y'' = 49C_1 e^{7x} + 9C_2 e^{-3x} \end{cases}$$

$$\begin{aligned} LS &= y'' - 4y' - 21y \\ &= (49C_1 e^{7x} + 9C_2 e^{-3x}) - 4(7C_1 e^{7x} - 3C_2 e^{-3x}) \\ &\quad - 21(C_1 e^{7x} + C_2 e^{-3x}) \\ &= 0C_1 e^{7x} + 0C_2 e^{-3x} \end{aligned}$$

← coefficient = 0 because 7 is a root of the auxiliary equation

$$= 0$$

$$= RS \quad \checkmark$$

Ex: Solve $2y'' - 24y' + 72y = 0$

$2m^2 - 24m + 72 = 0$

$2(m^2 - 12m + 36) = 0$

$2(m-6)^2 = 0$

$m = 6, 6$

$y_1 = e^{6x} \quad y_2 = xe^{6x}$

General Solution

$$\boxed{y = C_1 e^{6x} + C_2 x e^{6x}}$$

or $$\boxed{y = (C_1 + C_2 x) e^{6x}}$$

Ex: Solve $y'' - 6y' + 13y = 0$

$m^2 - 6m + 13 = 0$

$$m = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 13}}{2}$$

$$m = \frac{6 \pm \sqrt{-16}}{2} \qquad \sqrt{16}\sqrt{-1}$$

$$m = \frac{6 \pm 4i}{2}$$

$$m = 3 \pm 2i$$

$$y_1 = e^{3x}\cos 2x \qquad y_2 = e^{3x}\sin 2x$$

General Solution $y = C_1 e^{3x}\cos 2x + C_2 e^{3x}\sin 2x$

or $y = e^{3x}[C_1\cos 2x + C_2 \sin 2x]$

Ex: Solve $y'' + 49y = 0$, $y(0) = -5$, $y'(0) = 21$

$$m^2 + 49 = 0$$

$$m^2 = -49$$

$$m = \pm\sqrt{-49} \quad \sqrt{49}\sqrt{-1}$$

$$m = \pm 7i$$

$$m = 0 \pm 7i$$

General Solution $y = e^{0x}[C_1\cos 7x + C_2 \sin 7x]$

$$\boxed{y = C_1\cos 7x + C_2 \sin 7x}$$

Sub $x=0$, $y=-5$: $-5 = C_1$

$$\boxed{y = -5\cos 7x + C_2 \sin 7x}$$

$$y' = 35\sin 7x + 7C_2 \cos 7x$$

Sub $x=0$, $y'=21$: $21 = 0 + 7C_2$

$$C_2 = 3$$

$$\boxed{y = -5\cos 7x + 3\sin 7x}$$

EX: Solve $y^{(4)} + 8y'' + 16y = 0$

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2)^2 + 8m^2 + 16 = 0$$

$$(m^2+4)^2 = 0$$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4} \quad \sqrt{4}\sqrt{-1}$$

$$m = \pm 2i$$

$$\boxed{m = 0 \pm 2i,\ 0 \pm 2i}$$

$y_1 = e^{0x}\cos 2x = \cos 2x$

$y_2 = e^{0x}\sin 2x = \sin 2x$

$y_3 = x\cos 2x$

$y_4 = x\sin 2x$

General solution $y = C_1\cos 2x + C_2\sin 2x + C_3 x\cos 2x + C_4 x\sin 2x$

or $\boxed{y = (C_1 + C_3 x)\cos 2x + (C_2 + C_4 x)\sin 2x}$

Ex: a) If $m = 1, 3, 3$

$y_1 = e^x \quad y_2 = e^{3x} \quad y_3 = xe^{3x}$

$y = C_1 e^x + C_2 e^{3x} + C_3 x e^{3x}$

or $y = C_1 e^x + (C_2 + C_3 x)e^{3x}$

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b) If $m = 4, 4, 3 \pm 5i$

$y_1 = e^{4x} \quad y_2 = xe^{4x} \quad y_3 = e^{3x}\cos 5x$

$y_4 = e^{3x}\sin 5x$

$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 e^{3x}\cos 5x + C_4 e^{3x}\sin 5x$

or $y = (C_1 + C_2 x)e^{4x} + e^{3x}(C_3\cos 5x + C_4\sin 5x)$

Rational Roots Theorem:

Any rational roots of a polynomial are of the form $\frac{p}{q}$, where p divides the constant and q divides the leading coefficient.

Ex: Solve $2y''' - 7y'' + 7y' - 2y = 0$

$y(0)=3 \quad y'(0)=5 \quad y''(0)=9$

$2m^3 - 7m^2 + 7m - 2 = 0$

Possibilities for p: $\pm 1, \pm 2$

q: $\pm 1, \pm 2$

$\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm 2$

Possible roots

Test $m=-1$: $2(-1)^3 - 7(-1)^2 + 7(-1) - 2 = 0$?

No

Test $m=1$: $2(1)^3 - 7(1)^2 + 7(1) - 2 = 0$?

YES

$$2m^3 - 7m^2 + 7m - 2 = 0$$

$$(m-1)(\ ?\ ) = 0$$

$m=1$ is a root

Create correct first and last terms:

$$(m-1)(2m^2 + ?m + 2) = 0$$

Middle coefficient must give $-7m^2$ upon expansion:

$$(m-1)(2m^2 - 5m + 2) = 0$$

$m = 1$

$$2m^2 - 5m + 2 = 0$$

$$m = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(2)}}{4}$$

$$m = \frac{5 \pm \sqrt{9}}{4}$$

$$m = 2, \frac{1}{2}$$

$$\boxed{m = 1, 2, \tfrac{1}{2}}$$

$$\boxed{y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{x/2}}$$

$x=0$, $y=3$: $3 = C_1 + C_2 + C_3$ (1)

$$y' = C_1 e^x + 2C_2 e^{2x} + \frac{C_3}{2} e^{x/2}$$

$x=0$, $y'=5$: $5 = C_1 + 2C_2 + \frac{C_3}{2}$ (2)

$$y'' = C_1 e^x + 4C_2 e^{2x} + \frac{C_3}{4} e^{x/2}$$

$x=0$, $y''=9$: $9 = C_1 + 4C_2 + \frac{C_3}{4}$ (3)

$$\begin{array}{ccc|c} C_1 & C_2 & C_3 & \\ \hline 1 & 1 & 1 & 3 \\ 1 & 2 & 1/2 & 5 \\ 1 & 4 & 1/4 & 9 \end{array}$$

Gauss-Jordan ⇝

$$\begin{array}{ccc|c} C_1 & C_2 & C_3 & \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}$$

$C_1 = 1 \quad C_2 = 2 \quad C_3 = 0$

$$\boxed{y = e^x + 2e^{2x}}$$