4.2 Finding a Second Solution

Standard form for a homogeneous 2nd order linear DE:

$$y'' + P(x)y' + Q(x)y = 0$$

Given one solution y_1,
a 2nd linearly independent solution is:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

General solution: $y = C_1 y_1 + C_2 y_2$

Ex: $y'' = 6y' - 9y$

a) Check that $y_1 = e^{3x}$ is a solution

$$\begin{cases} y_1 = e^{3x} \\ y_1' = 3e^{3x} \\ y_1'' = 9e^{3x} \end{cases}$$

$LS = y'' = 9e^{3x}$

$RS = 6y' - 9y$
$= 18e^{3x} - 9e^{3x}$
$= 9e^{3x}$

✓

b) Find a 2nd linearly independent solution

$$\boxed{y'' - 6y' + 9y = 0}$$

$P(x) = -6$

$\int P(x)\,dx = -6x$ ← don't use a constant here

$-\int P(x)\,dx = 6x$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= e^{3x} \int \frac{e^{6x}}{(e^{3x})^2}\,dx$$

$$= e^{3x} \int \frac{e^{6x}}{e^{6x}}\,dx$$

$$= e^{3x} \int 1\,dx$$

$$= e^{3x}(x)$$

$$= xe^{3x}$$

c) General solution of $y'' - 6y' + 9y = 0$?

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{3x} + C_2 x e^{3x}$$

Ex: $y'' + 25y = 0$ has solution

$y_1 = \sin 5x$

a) Find a 2nd linearly independent solution

$$P(x) = 0$$

$$\int P(x)\,dx = 0$$

$$-\int P(x)\,dx = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= \sin 5x \int \frac{e^0}{(\sin 5x)^2}\,dx$$

$$= \sin 5x \int \frac{1}{\sin^2 5x}\,dx$$

$$= \sin 5x \int \csc^2 5x\,dx$$

$$= \sin 5x \left[-\frac{1}{5}\cot 5x\right]$$

$$= -\frac{1}{5}\sin 5x \cdot \frac{\cos 5x}{\sin 5x}$$

$$= -\frac{1}{5}\cos 5x$$

Simpler: $y_2 = \cos 5x$ since any multiple of y_2 also solves the DE.

b) General solution to DE?

$y = C_1 y_1 + C_2 y_2$

$y = C_1 \sin 5x + C_2 \cos 5x$

Ex: $y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0$ has solution $y_1 = x^2$

a) Find a 2nd linearly independent solution

$$P(x) = \frac{2}{x}$$

$$\int P(x)\,dx = 2\ln|x|$$

$$-\int P(x)\,dx = -2\ln|x|$$

$$e^{-\int P(x)\,dx} = e^{-2\ln|x|}$$

$$= e^{\ln|x|^{-2}}$$

$$= |x|^{-2}$$

$$= x^{-2}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= x^2 \int \frac{x^{-2}}{(x^2)^2}\,dx$$

$$= x^2 \int x^{-6}\,dx$$

$$= x^2 \left(\frac{-1}{5} x^{-5}\right)$$

$$= -\frac{1}{5} x^{-3}$$

Simpler: $y_2 = x^{-3}$

b) General solution to DE?

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 x^2 + C_2 x^{-3}$$