4.1 Intro to Higher-Order Linear DE's

n^{th} order linear DE:

$$a_n(x)\cdot y^{(n)} + \dots + a_1(x)\cdot y' + a_0(x)\, y = g(x)$$

Coefficients are functions of x (don't involve y)

The DE is **homogeneous** if $g(x) = 0$

Ex: a) $y'' - 9y = 0$

is a homogeneous 2nd order linear DE

b) $y'' - 9y = x$

is a non-homogeneous 2nd order linear DE

NOT TO BE CONFUSED WITH:

$$x^2\,dx + (3x^2 - y^2)\,dy = 0$$

"homogeneous of degree 2"

Ex: $y'' - 9y = 0$

has general solution $y = C_1 e^{3x} + C_2 e^{-3x}$

C_1, C_2: any real numbers

describes all possible solutions

Some particular solutions:

$y = 0$

$y = -2e^{3x}$

$y = 4e^{-3x}$

$y = \sqrt{2}e^{3x} + \sqrt{5}e^{-3x}$

Functions $f(x)$, $g(x)$ and $h(x)$ are linearly dependent exactly when

$$\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = 0$$

← "Wronskian"

Ex: a) $f(x)=\sin^2 x$, $g(x)=\cos^2 x$ and $h(x)=7$ are linearly dependent because

$$\begin{vmatrix} \sin^2 x & \cos^2 x & 7 \\ 2\sin x\cos x & -2\cos x\sin x & 0 \\ f''(x) & -f''(x) & 0 \end{vmatrix} = 7\left[-2\sin x\cos x \cdot f''(x) + 2\cos x\sin x\, f''(x)\right] = 0$$

b) $f(x)=e^{2x}$, $g(x)=e^{3x}$ and $h(x)=1$ are linearly independent because

$$\begin{vmatrix} e^{2x} & e^{3x} & 1 \\ 2e^{2x} & 3e^{3x} & 0 \\ 4e^{2x} & 9e^{3x} & 0 \end{vmatrix}$$

$$= 1\left[18e^{5x} - 12e^{5x}\right]$$
$$= 6e^{5x}$$
$$\neq 0$$

To solve a homogeneous n^{th} order linear DE:

Find n linearly independent solutions

$$y_1, y_2, \cdots, y_n$$

General solution is $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$

Ex: $y''' - 11y'' + 30y' = 0$ has

3 linearly independent solutions

$$y_1 = e^{5x} \quad y_2 = e^{6x} \quad y_3 = 1$$

General solution?

hom. 3rd order linear DE

$$y = C_1y_1 + C_2y_2 + C_3y_3$$

$$\boxed{y = C_1e^{5x} + C_2e^{6x} + C_3}$$