3.1 Linear Models

Recall: A is proportional to B means
$A = kB$ k: constant

Ex: Rate of population increase is proportional to population size. Population doubles in 5 years. Time to triple?

P = population size t = time (years)
P_0 = initial population

rate $\rightarrow \frac{dP}{dt} = kP$ k: constant

$$\frac{dP}{P} = k\,dt$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$$|P| = e^{kt+C_1}$$

$$P = \pm e^{kt} \cdot e^{C_1}$$

$$\boxed{P = Ce^{kt}}$$

Sub $t=0$, $P=P_0$: $P_0 = C(1)$
$C = P_0$

$$\boxed{P = P_0 e^{kt}}$$

Sub $t=5$, $P=2P_0$: $\quad 2P_0 = P_0 e^{5k}$

$$2 = e^{5k}$$

$$\ln 2 = 5k$$

$$\tfrac{1}{5}\ln 2 = k$$

$$\boxed{P = P_0 e^{\frac{1}{5}\ln 2\, t}}$$

Sub $P=3P_0$: $\quad 3P_0 = P_0 e^{\frac{1}{5}\ln 2\, t}$

$$3 = e^{\frac{1}{5}\ln 2\, t}$$

$$\ln 3 = \tfrac{1}{5}\ln 2\, t$$

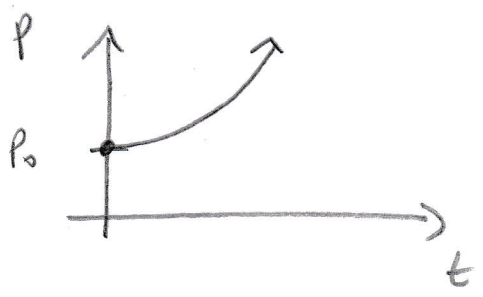
$$\frac{5\ln 3}{\ln 2} = t$$

$$t \approx 8 \text{ years}$$

Notice: We did not need to know P_0, but it was handy to have a symbol for it.

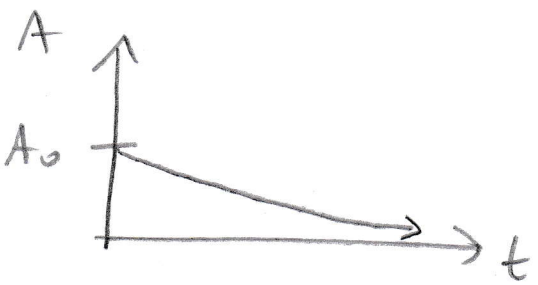
$P = P_0 e^{kt}$ with $k>0$ is exponential growth

$P \to \infty$ as $t \to \infty$



$A = A_0 e^{kt}$ with $k<0$ is exponential decay

$A \to 0$ as $t \to \infty$



Ex: Rate of decay of a radioactive substance is proportional to amount remaining. After 1000 years, 3.5% of original amount has decayed. Find the substance's half-life.

A = amount remaining t = time (years)

A_0 = initial amount Half-life: time for 50% to decay

rate → $\frac{dA}{dt} = kA$

$\frac{dA}{A} = k\,dt$

$\int \frac{dA}{A} = \int k\,dt$

$\ln|A| = kt + C_1$

$|A| = e^{kt+C_1}$

$A = \pm e^{C_1} \cdot e^{kt}$

$\boxed{A = Ce^{kt}}$

Sub $t=0$, $A=A_0$: $A_0 = C(1)$

$C = A_0$

$\boxed{A = A_0 e^{kt}}$

$t = 1000$

96.5% remaining : $A = 0.965 A_0$

$$0.965 A_0 = A_0 e^{1000k}$$

$$0.965 = e^{1000k}$$

$$\ln 0.965 = 1000k$$

$$\frac{1}{1000} \ln 0.965 = k$$

$$\boxed{A = A_0 e^{\frac{1}{1000} \ln 0.965 t}}$$

Half-life $A = 0.5 A_0$:

$$0.5 A_0 = A_0 e^{\frac{1}{1000} \ln 0.965 t}$$

$$0.5 = e^{\frac{1}{1000} \ln 0.965 t}$$

$$\ln 0.5 = \frac{1}{1000} \ln 0.965 t$$

$$\frac{1000 \ln 0.5}{\ln 0.965} = t$$

$$t \approx 19{,}000 \text{ years}$$

Newton's Law of Cooling

Rate at which an object warms/cools is proportional to temperature difference between it and its environment.

T = object's temp. T_m = ambient temp. (constant) t = time

$$\boxed{\frac{dT}{dt} = k(T - T_m)}$$

Ex: Room temp = 20°C
Initially coffee is 90°C. After 40 mins coffee is 30°C. Time when coffee is 60°C?

T = coffee temp (°C) t = time (mins)

$$\frac{dT}{dt} = k(T-20) \quad k: \text{constant}$$

$$\frac{dT}{T-20} = k\,dt$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln|T-20| = kt + C_1$$

$$|T-20| = e^{kt+C_1}$$

$$T-20 = \pm e^{C_1} e^{kt}$$

$$T-20 = Ce^{kt}$$

Sub $T=90$, $t=0$: $70 = C$

$$\boxed{T = 20 + 70e^{kt}}$$

$T=30$, $t=40$: $30 = 20 + 70e^{40k}$

$10 = 70e^{40k}$

$\frac{1}{7} = e^{40k}$

$\ln \frac{1}{7} = 40k$

$\frac{1}{40}\ln \frac{1}{7} = k$

$$\boxed{T = 20 + 70e^{\frac{1}{40}(\ln \frac{1}{7})t}}$$

$T=60$: $60 = 20 + 70e^{\frac{1}{40}(\ln \frac{1}{7})t}$

$40 = 70e^{\frac{1}{40}(\ln \frac{1}{7})t}$

$\frac{4}{7} = e^{\frac{1}{40}(\ln \frac{1}{7})t}$

$\ln \frac{4}{7} = \frac{1}{40}(\ln \frac{1}{7})t$

$$\frac{40 \ln \frac{4}{7}}{\ln \frac{1}{7}} = t$$

$t \approx 12$ minutes

Ex: A tank initially contains 500L of water. A solution of 2kg salt/L flows in at 5L/min. Contents are mixed, and flow out at 10L/min. Find a formula for mass of salt in the tank.

Losing 5L of solution every minute

A = mass of salt (kg) t = time (mins)

A(0) = 0 Volume = 500 − 5t

Inflow rate = $2\text{kg/L} \cdot 5\text{L/min} = \boxed{10\text{kg/min}}$

Outflow rate = $\frac{A\text{ kg}}{500-5t\text{ L}} \cdot 10\text{ L/min}$

$= \frac{10A}{500-5t}$ kg/min

$= \boxed{\frac{2A}{100-t}\text{ kg/min}}$

$$\frac{dA}{dt} = \text{Inflow rate} - \text{Outflow rate}$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t}$$

$$\boxed{\frac{dA}{dt} + \frac{2}{100-t}A = 10}$$ Linear

$$P(t) = \frac{2}{100-t}$$

I.F. $e^{\int \frac{2}{100-t}dt}$ $\rightarrow$

$= e^{-2\ln|100-t|}$

$= e^{\ln(100-t)^{-2}}$

$= (100-t)^{-2}$

Interval: $0 < t < 100$

$u = 100-t$

$du = -dt$

$dt = -du$

$\int \frac{2}{100-t}dt = \int \frac{-2du}{u}$

$= -2\ln|u| + C_1$

$$(100-t)^{-2}\frac{dA}{dt} + 2(100-t)^{-3}A = 10(100-t)^{-2}$$

Integrate both sides:

$$(100-t)^{-2}A = 10(100-t)^{-1} + C$$

$\int 10(100-t)^{-2}dt$

$= \int -10u^{-2}du$

$= 10u^{-1} + C$

$= 10(100-t)^{-1} + C$

$$\boxed{A = 10(100-t) + C(100-t)^2}$$

$t=0$, $A=0$: $\quad 0 = 10(100) + C(100)^2$

$$C = -0.1$$

$$\boxed{A = 10(100-t) - 0.1(100-t)^2}$$

or

$$A = 1000 - 10t - 0.1(10000 - 200t + t^2)$$

$$\boxed{A = 10t - 0.1t^2}$$

Interval : $0 < t < 100$

In fact, formula works for $0 \leq t \leq 100$:

$A(0) = 0$ pure water ✓

$A(100) = 0$ empty tank ✓