2.5 Solutions by Substitution

Ex: $(y^2+yx)dx + x^2dy = 0$

↑ ↑ ↑

each term has degree 2

DE is "homogeneous of degree 2"

Ex: $(3x^2y - y^3)dx - 7x^3dy = 0$

"homogeneous of degree 3"

Sub $\begin{cases} y = ux \\ dy = udx + xdu \end{cases}$ or $\begin{cases} x = vy \\ dx = vdy + ydv \end{cases}$

into a homogeneous DE

to get a separable DE

Sub $y = ux$ when the dy term is simpler than the dx-term.

Ex: Solve explicitly

$(x^2+y^2)dx - xy\,dy = 0$

Homogeneous (of degree 2)

dy term is simpler

Sub $\begin{cases} y = ux \\ dy = udx + xdu \end{cases}$

$$(x^2 + u^2x^2)dx - x(ux)(u\,dx + x\,du) = 0$$

$$x^2\,dx + u^2x^2\,dx - ux^2(u\,dx + x\,du) = 0$$

$$x^2\,dx + u^2x^2\,dx - u^2x^2\,dx - ux^3\,du = 0$$

$$x^2\,dx - ux^3\,du = 0 \qquad \text{SEPARABLE}$$

$$x^2\,dx = ux^3\,du$$

$$\frac{1}{x}\,dx = u\,du$$

$$\int \frac{1}{x}\,dx = \int u\,du$$

$$\ln|x| = \frac{u^2}{2} + C_1$$

$y = ux$

$u = \frac{y}{x}$:

$$\ln|x| = \frac{1}{2}\,\frac{y^2}{x^2} + C_1$$

$$2\ln|x| = \frac{y^2}{x^2} + C_2$$

$$2\ln|x| + C_3 = \frac{y^2}{x^2}$$

$$\pm\sqrt{2\ln|x| + C_3} = \frac{y}{x}$$

$$y = \pm x\sqrt{2\ln|x| + C_3}$$

Ex: Solve $(x + ye^{y/x})dx - xe^{y/x}dy = 0$, $y(1) = 0$

Note: Ignore e^x, $\sin x$, $\ln x$ etc. when calculating degree
$\sqrt{x} = x^{1/2}$ has degree $\frac{1}{2}$

Homogeneous (of degree 1)

dy term is simple

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$ $\quad y/x = u$

$$(x + uxe^u)dx - xe^u(u\,dx + x\,du) = 0$$

$$x\,dx + uxe^u dx - uxe^u dx - x^2 e^u du = 0$$

$$x\,dx - x^2 e^u du = 0 \qquad \text{SEPARABLE}$$

$$x\,dx = x^2 e^u du$$

$$\frac{1}{x}dx = e^u du$$

$$\int \frac{1}{x}dx = \int e^u du$$

$$\ln|x| = e^u + C_1$$

$u = y/x$: $\quad \ln|x| = e^{y/x} + C_1$

Sub $x = 1$, $y = 0$: $\quad 0 = 1 + C_1$

$C_1 = -1$

$$\boxed{\ln|x| = e^{y/x} - 1}$$

Bernoulli DE

$$\frac{dy}{dx} + Q(x)y = f(x)y^n$$

Like linear, but with y^n

When $n \neq 0, 1$

Sub $y = u^{\frac{1}{1-n}}$ to get a linear DE

Note: When $n=0$, the DE is linear
" $n=1$ " separable

Ex: Solve explicitly

$$\frac{1}{y}\frac{dy}{dx} = e^x y + 1$$

$$\frac{dy}{dx} = e^x y^2 + y$$

$$\boxed{\frac{dy}{dx} - y = e^x y^2}$$ Standard Form

Bernoulli $n=2$

Sub $y = u^{\frac{1}{1-n}} = u^{\frac{1}{1-2}} = u^{-1}$

$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^x u^{-2}$$

Mult. by $-u^2$:

$$\boxed{\frac{du}{dx} + u = -e^x}$$

LINEAR

$P(x) = 1$

I.F. $e^{\int P(x)dx} = e^{\int 1\,dx}$

$= e^x$

$$e^x \frac{du}{dx} + e^x u = -e^{2x}$$

Integrate both sides:

$$e^x u = -\frac{1}{2}e^{2x} + C_1$$

$y = u^{-1}$

$u = y^{-1}$: $e^x y^{-1} = -\frac{1}{2}e^{2x} + C_1$

$$\frac{e^x}{C_1 - \frac{1}{2}e^{2x}} = y$$

or $y = \dfrac{2e^x}{C_2 - e^{2x}}$

To solve $\frac{dy}{dx} = f(Ax+By+C)$

Sub $u = Ax+By+C$
to get a separable DE

Ex: Solve explicitly

$$\frac{dy}{dx} = (x+y+5)^2$$

Sub $\begin{cases} u = x+y+5 \\ \frac{du}{dx} = 1 + \frac{dy}{dx} \end{cases}$ $\qquad \frac{dy}{dx} = \frac{du}{dx} - 1$

$$\frac{du}{dx} - 1 = u^2 \qquad \text{SEPARABLE}$$

$$\frac{du}{dx} = u^2 + 1$$

$$\frac{du}{u^2+1} = dx$$

$$\int \frac{du}{u^2+1} = \int dx$$

$$\arctan u = x + C$$

$u = x+y+5$:

$$\arctan(x+y+5) = x + C$$

$$x+y+5 = \tan(x+C)$$

$$\boxed{y = \tan(x+C) - x - 5}$$

Recap:

DE	Substitution
Homogeneous	$y = ux$ or $x = vy$
Bernoulli $\frac{dy}{dx} + Q(x)y = f(x)y^n$	$y = u^{\frac{1}{1-n}}$
$\frac{dy}{dx} = f(Ax+By+C)$	$u = Ax + By + C$