2.4 Exact DE's

Let $f = x^3y + \sin(3x+y)$. Partial derivatives:

$\frac{\partial f}{\partial x}$ or $f_x = 3x^2y + 3\cos(3x+y)$

$\frac{\partial f}{\partial y}$ or $f_y = x^3 + \cos(3x+y)$

The differential of f: $df = f_x \cdot dx + f_y \cdot dy$

Here $df = (3x^2y + 3\cos(3x+y))dx + (x^3 + \cos(3x+y))dy$

A DE $M dx + N dy = 0$ is exact if the left side is a differential.

Quick way to check:
A DE is exact if $M_y = N_x$

Ex: $(3x^2 + \sin y)dx + x\cos y\, dy = 0$

$M = 3x^2 + \sin y$ $\quad$ $N = x\cos y$

$M_y = \cos y$ $\quad$ $N_x = \cos y$

$M_y = N_x$ ⌐

The DE is exact
LS is a differential

Ex: $x^3 y\,dx + (x+1)\,dy = 0$

$M_y = x^3 \qquad N_x = 1$

$M_y \neq N_x$

DE is not exact

LS is not a differential

Ex: Solve $\sin y\,dx + (2y + x\cos y)\,dy = 0$

$M_y = \cos y = N_x$

DE is exact

DE: $df = 0$

$\int df = \int 0$

$\boxed{f = C}$

Find f: $f = \int \sin y\,dx$ and $f = \int (2y + x\cos y)\,dy$

$f = x\sin y + g(y)$ (g(y): y-terms) $\qquad f = y^2 + x\sin y + h(x)$ (h(x): x-terms)

Comparing: $g(y) = y^2$, $h(x) = 0$

$$f = x\sin y + y^2$$

$$f = c$$

$$\boxed{x\sin y + y^2 = c}$$

Ex: Solve explicitly

$$(1+y^2)dx + (2xy+2y)dy = 0, \quad y(0)=1$$

$M_y = 2y = N_x$

DE is exact

$df = 0$

$\int df = \int 0$

$\boxed{f = c}$

Find f: $f = \int (1+y^2)dx$ and $f = \int (2xy+2y)dy$

$f = x + xy^2 + g(y)$ $\qquad$ $f = xy^2 + y^2 + h(x)$

$$f = xy^2 + y^2 + x$$

$$f = c$$

$$\boxed{xy^2 + y^2 + x = C}$$

Sub $x=0$, $y=1$: $1 = c$

$$xy^2 + y^2 + x = 1$$

$$(x+1)y^2 = 1-x$$

$$y^2 = \frac{1-x}{x+1}$$

$$y = \pm\sqrt{\frac{1-x}{x+1}}$$

But $y(0)=1$ Choose (+)

$$\boxed{y = \sqrt{\frac{1-x}{x+1}}}$$

EX: Solve $(e^{x+y} + 2y)dy = -(e^{x+y} + 3x^2)dx$

$$(e^{x+y} + 3x^2)dx + (e^{x+y} + 2y)dy = 0$$

$$\boxed{(e^x \cdot e^y + 3x^2)dx + (e^x \cdot e^y + 2y)dy = 0}$$

$M_y = e^x e^y = N_x$

DE is exact

DE : $df = 0$

$\int df = \int 0$

$\boxed{f = C}$

Find f: $f=\int(e^x e^y+3x^2)\,dx$ and $f=\int(e^x e^y+2y)\,dy$

$f = e^x e^y + x^3 + g(y)$ $\quad\Big\}\quad f = e^x e^y + y^2 + h(x)$

$$f = e^x e^y + x^3 + y^2$$

$$f = C$$

$$\boxed{e^x e^y + x^3 + y^2 = C}$$

or $\boxed{e^{x+y} + x^3 + y^2 = C}$

If the DE is not exact, we can sometimes find an integrating factor.

If $\dfrac{My-Nx}{N}$ is a function of x (no y-terms)

use $\text{I.F.} = e^{\int \frac{My-Nx}{N}\,dx}$

If $\dfrac{Nx-My}{M}$ is a function of y (no x-terms)

use $\text{I.F.} = e^{\int \frac{Nx-My}{M}\,dy}$

Ex: Solve $y\,dx + (2x - ye^y)\,dy = 0$

$M_y = 1 \qquad N_x = 2$

DE is not exact

$$\frac{M_y - N_x}{N} = \frac{-1}{2x - ye^y}$$

$$\frac{N_x - M_y}{M} = \frac{1}{y} \quad \text{function of } y$$

I.F. $e^{\int \frac{1}{y}\,dy}$

$= e^{\ln|y|}$

$= |y|$

$= y, \quad \boxed{y > 0}$

Mult. DE by y:

$y^2\,dx + (2xy - y^2e^y)\,dy = 0$ is now exact

DE: $df = 0$

$\int df = \int 0$

$\boxed{f = C}$

Find f:

$f = \int y^2\,dx$ and $f = \int (2xy - y^2 e^y)\,dy$

$f = y^2x + g(y)$

	D	I
⊕	$-y^2$	e^y
⊖	$-2y$	e^y
⊕	-2	e^y
	0	e^y

$f = xy^2 - y^2e^y + 2ye^y - 2e^y + h(x)$

$f = xy^2 + e^y(-y^2 + 2y - 2)$

$f = C$

$$\boxed{xy^2 + e^y(-y^2 + 2y - 2) = C}$$

Ex: Solve $(e^{2x} + y - 1)\,dx - dy = 0$

$M_y = 1 \quad N_x = 0$

DE is not exact

$\frac{M_y - N_x}{N} = \frac{1}{-1} = -1$ function of x ☺

I.F. $e^{\int -1\,dx} = e^{-x}$. Multiply DE by e^{-x}:

$(e^x + ye^{-x} - e^{-x})\,dx - e^{-x}\,dy = 0$ is now exact

$df = 0$

$\int df = \int 0$

$\boxed{f = C}$

Find f:

$f = \int (e^x + ye^{-x} - e^{-x})\,dx$ and $f = \int (-e^{-x})\,dy$

$f = e^x - ye^{-x} + e^{-x} + g(y)$ $\qquad$ $f = -e^{-x}y + h(x)$

$g(y) = 0$
$h(x) = e^x + e^{-x}$

$f = -e^{-x}y + e^x + e^{-x}$

$f = c$

$$\boxed{-e^{-x}y + e^x + e^{-x} = c}$$

or $\quad -y + e^{2x} + 1 = ce^x$

or $\quad e^{2x} + C_1 e^x + 1 = y$

Methods Covered so far:

Separable
Linear
Exact

Ex: Identify the method(s)

a) $\frac{dy}{dx} - x^2 + 3x^2y = 0$

$\frac{dy}{dx} = x^2 - 3x^2y$

$dy = (x^2 - 3x^2y)\,dx$

$dy = x^2(1-3y)\,dx$

$\frac{dy}{1-3y} = x^2\,dx$

Separable

$\frac{dy}{dx} + 3x^2y = x^2$

Linear

★FASTEST★

$dy + (3x^2y - x^2)\,dx = 0$

$(3x^2y - x^2)\,dx + dy = 0$

$M_y = 3x^2 \quad N_x = 0$

Not Exact

(Could find I.F.)

b) $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

Not separable

Not linear

$M_y = 4xy = N_x$

Exact