2.3 Linear DE's

We'll solve $\boxed{\frac{dy}{dx} + P(x)y = f(x)}$

"1st order linear DE"

Recall: $\frac{d}{dx}[e^{x^2}y] = e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y$

$$\int (e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y)\,dx = e^{x^2}y + \text{constant}$$

Ex: Solve explicitly (solve for y)

$$y' + 2xy - x = 0$$

1) $\boxed{\frac{dy}{dx} + 2xy = x}$ Standard Form

2) $P(x) = 2x$

3) Integrating Factor

$$e^{\int P(x)dx}$$

$$= e^{\int 2x\,dx}$$

$$= e^{x^2}$$ ← Don't use a constant here

4) Multiply 1) by the integrating factor

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

5) $\int\left(e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y\right)dx = \int xe^{x^2}dx$

Comes from product rule

$u = x^2$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

$\int \frac{1}{2}e^u du$

$= \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2} + C$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C$$

6) Mult. by e^{-x^2}

$$\boxed{y = \frac{1}{2} + Ce^{-x^2}}$$

Shortcut for step 5):

$$\int\left(e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y\right)dx = e^{x^2}y$$

Ex: Solve explicitly

$$x\frac{dy}{dx} - y - 2x^2 = 0, \quad y(1) = 5$$

1) $x\frac{dy}{dx} - y = 2x^2$

$$\boxed{\frac{dy}{dx} - \frac{1}{x}y = 2x}$$

2) $P(x) = -\frac{1}{x}$

3) $e^{\int P(x)dx}$

$= e^{\int -\frac{1}{x}dx}$

$= e^{-\ln|x|}$

$= e^{\ln|x|^{-1}}$

$= \frac{1}{|x|}$

$= \begin{cases} \frac{1}{x}, & x > 0 \\ -\frac{1}{x}, & x < 0 \end{cases}$

$= \frac{1}{x}$ since $x = 1$ was given

$\boxed{\text{interval: } x > 0}$

4) $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2$

5) Integrate with respect to x
LS comes from product rule (use shortcut)

$$\frac{1}{x}y = 2x + C$$

6) $$y = 2x^2 + Cx$$

Sub $x=1$, $y=5$: $5 = 2 + C$

$C = 3$

$$y = 2x^2 + 3x$$

Interval: $x > 0$

Ex: Solve explicitly

$$(x^2-16)\frac{dy}{dx} + xy = x$$

1) $$\boxed{\frac{dy}{dx} + \frac{x}{x^2-16}y = \frac{x}{x^2-16}}$$

2) $$P(x) = \frac{x}{x^2-16}$$

3) $$\text{I.F.} = e^{\int \frac{x}{x^2-16}\,dx}$$

$$= e^{\frac{1}{2}\ln|x^2-16|}$$

$$= e^{\ln\sqrt{x^2-16}}$$

$$= \sqrt{x^2-16}$$

$u = x^2-16$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}du$
$\int \frac{1}{2}\frac{du}{u}$
$= \frac{1}{2}\ln|u|$
$= \frac{1}{2}\ln|x^2-16|$

4) $$\sqrt{x^2-16}\frac{dy}{dx} + \frac{x}{\sqrt{x^2-16}}y = \frac{x}{\sqrt{x^2-16}}$$

5) Integrate with respect to x:

$$\sqrt{x^2-16}\,y = \sqrt{x^2-16} + C$$

6) $$\boxed{y = 1 + \frac{C}{\sqrt{x^2-16}}}$$

$u = x^2-16$
$du = 2x\,dx$
$x\,dx = \frac{1}{2}du$
$\int \frac{x\,dx}{\sqrt{x^2-16}} = \int \frac{1}{2}\frac{du}{\sqrt{u}}$
$= \sqrt{u} + C$
$= \sqrt{x^2-16} + C$

Note: $\frac{C}{\sqrt{x^2-16}}$ is a transient term: it $\to 0$ as $x \to \infty$

As $x \to \infty$, $y \to 1$

Ex: Solve explicitly:

$$t\frac{dy}{dt} + 3y = t+1 \quad , \quad y(-1) = \frac{1}{3}$$

1) $\boxed{\frac{dy}{dt} + \frac{3}{t}y = 1 + \frac{1}{t}}$

2) $P(t) = \frac{3}{t}$

3) $e^{\int P(t)dt} = e^{\int \frac{3}{t}dt}$

$= e^{3\ln|t|}$

$= e^{\ln|t|^3}$

$= |t|^3$

$= \begin{cases} t^3, & t>0 \\ -t^3, & t<0 \end{cases}$

$= -t^3$ since we're given $t=-1$

$\boxed{t<0}$

4) $-t^3\frac{dy}{dt} - 3t^2y = -t^3 - t^2$

5) Integrate with respect to t:

$$-t^3y = -\frac{t^4}{4} - \frac{t^3}{3} + C$$

6) $\boxed{y = \frac{t}{4} + \frac{1}{3} - \frac{c}{t^3}}$

Sub $y = \frac{1}{3}$, $t = -1$: $\frac{1}{3} = \frac{-1}{4} + \frac{1}{3} - \frac{c}{(-1)}$

$0 = -\frac{1}{4} + c$

$c = \frac{1}{4}$

$$\boxed{y = \frac{t}{4} + \frac{1}{3} - \frac{1}{4t^3}}$$

Interval: $t < 0$

Ex: Solve explicitly

$$y\,dx - (x + y^6)\,dy = 0$$

NOT LINEAR IN y: $-(x+y^6)\,dy + y\,dx = 0$

$$dy - \frac{y}{x+y^6}\,dx = 0$$

$$\frac{dy}{dx} - \frac{y}{x+y^6} = 0$$

NOT $\frac{dy}{dx} + P(x)y = f(x)$

IT IS LINEAR IN x:

$$y\,dx - (x+y^6)\,dy = 0$$

$$dx - \frac{(x+y^6)}{y}\,dy = 0$$

$$\frac{dx}{dy} - \frac{x+y^6}{y} = 0$$

$$\frac{dx}{dy} - \frac{x}{y} - y^5 = 0$$

1) $$\boxed{\frac{dx}{dy} - \frac{1}{y}\cdot x = y^5}$$

2) $P(y) = -\frac{1}{y}$

3) $$e^{\int -\frac{1}{y}\,dy} = e^{-\ln|y|}$$

$$= e^{\ln \frac{1}{|y|}}$$

$$= \frac{1}{|y|}$$

$$= \begin{cases} \frac{1}{y}, & y>0 \\ -\frac{1}{y}, & y<0 \end{cases} \quad \text{(Choose one)}$$

$$= \boxed{\frac{1}{y},\ y>0}$$

4) $$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = y^4$$

5) Integrate <u>with respect to y</u>:

$$\frac{1}{y}x = \frac{y^5}{5} + C$$

6)

$$x = \frac{y^6}{5} + Cy$$

Interval: $y > 0$