2.2 Separable Variables

Ex: Solve $\frac{dy}{dx} = 2xy^2$

$$\frac{dy}{y^2} = 2x\,dx$$

Variables are separated
"Separable DE"

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$-y^{-1} = x^2 + C$$

Note: $-y^{-1} + C_1 = x^2 + C_2$ is unnecessary

$$-y^{-1} = x^2 + \underbrace{C_2 - C_1}_{C}$$

$$-\frac{1}{y} = x^2 + C$$

$$\frac{-1}{x^2+C} = y$$

$$\boxed{y = \frac{-1}{x^2+C}}$$

Ex: Solve $\frac{dy}{dx} = \frac{y}{x}$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C_1$$

$$e^{\ln|y|} = e^{\ln|x| + C_1}$$

$$e^{\ln|y|} = e^{\ln|x|} \cdot e^{C_1}$$

$$|y| = |x| \cdot e^{C_1}$$

$$y = \pm e^{C_1} x$$

$$\boxed{y = Cx}$$

Ex: Solve $\frac{dy}{dx} = \frac{y}{2-x}$ if $y(0) = 6$

Interval of solution?

$$\frac{dy}{y} = \frac{dx}{2-x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{2-x}$$

$$\ln|y| = -\ln|2-x| + C_1$$

$$e^{\ln|y|} = e^{-\ln|2-x| + C_1}$$

$u = 2-x$
$du = -dx$
$dx = -du$
$\int \frac{-du}{u} = -\ln|u| + C_1$
$= -\ln|2-x| + C_1$

$$e^{\ln|y|} = e^{-\ln|2-x|} \cdot e^{C_1}$$

$$e^{\ln|y|} = e^{\ln|2-x|^{-1}} \cdot e^{C_1}$$

$$|y| = |2-x|^{-1} \cdot e^{C_1}$$

$$y = \pm \frac{e^{C_1}}{2-x}$$

$$\boxed{y = \frac{C}{2-x}}$$

Sub $x=0$, $y=6$: $6 = \frac{C}{2}$

$C = 12$

$$\boxed{y = \frac{12}{2-x}}$$

Interval of Solution? $\{x \mid y \text{ is continuous}\}$

$x < 2$ or $x > 2$

But $x=0$ was given

$$\boxed{x < 2}$$

Ex: Solve $x\,dx + y\,dy = 0$ with $y(2) = -1$

$$x\,dx = -y\,dy$$

$$\int x\,dx = \int -y\,dy$$

$$\frac{x^2}{2} = \frac{-y^2}{2} + C_1$$

$$x^2 = -y^2 + \cancel{2C_1}\ C$$

Sub $x = 2$, $y = -1$: $4 = -1 + C$

$$C = 5$$

$$x^2 = -y^2 + 5$$

$$y^2 = 5 - x^2$$

$$y = \pm\sqrt{5 - x^2}$$

But $y(2) = -1$ Choose $-$

$$\boxed{y = -\sqrt{5 - x^2}}$$

Ex: Solve $\frac{dy}{dx} = y^2 - 9$

$$\frac{dy}{y^2-9} = dx$$

$$\int \frac{dy}{y^2-9} = \int dx$$

Partial Fractions for $\frac{1}{y^2-9}$

$$\frac{1}{(y-3)(y+3)} = \frac{A}{y-3} + \frac{B}{y+3}$$

$$\boxed{1 = A(y+3) + B(y-3)}$$

Sub $y=-3$: $1 = B(-6)$

$B = \frac{-1}{6}$

Sub $y=3$: $1 = A(6)$

$A = \frac{1}{6}$

$$\frac{1}{y^2-9} = \frac{1}{6} \cdot \frac{1}{y-3} - \frac{1}{6} \cdot \frac{1}{y+3}$$

$$\int \left(\frac{1}{6} \cdot \frac{1}{y-3} - \frac{1}{6} \cdot \frac{1}{y+3} \right) dy = \int dx$$

$$\frac{1}{6} \ln|y-3| - \frac{1}{6} \ln|y+3| = x + C_1$$

$$\ln|y-3| - \ln|y+3| = 6x + C_2$$

$$\ln \frac{|y-3|}{|y+3|} = 6x + C_2$$

$$\frac{|y-3|}{|y+3|} = e^{6x+C_2}$$

$$\frac{y-3}{y+3} = \pm e^{C_2} \cdot e^{6x}$$

$$\frac{y-3}{y+3} = Ce^{6x}$$

$$y-3 = Ce^{6x}(y+3)$$

$$y-3 = Ce^{6x}y + 3Ce^{6x}$$

$$y - Ce^{6x}y = 3 + 3Ce^{6x}$$

$$y(1-Ce^{6x}) = 3(1+Ce^{6x})$$

$$y = \frac{3(1+Ce^{6x})}{1-Ce^{6x}}$$

$y-3 = Ce^{6x}(y+3)$ IMPLICIT SOLUTION

$y = \frac{3(1+Ce^{6x})}{1-Ce^{6x}}$ EXPLICIT SOLUTION

<u>Standard form</u> for implicit solutions:
Get rid of ln, absolute value, clear fractions

Ex: Find an implicit solution

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)}$$

$$\frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx$$

$$\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$$

Long Division

$$\begin{array}{r} 1 \\ y+3 \overline{\big) \; y-2} \\ -(y+3) \\ \hline -5 \end{array}$$

$$\boxed{\frac{y-2}{y+3} = 1 - \frac{5}{y+3}}$$

$$\begin{array}{r} 1 \\ x+4 \overline{\big) \; x-1} \\ -(x+4) \\ \hline -5 \end{array}$$

$$\boxed{\frac{x-1}{x+4} = 1 - \frac{5}{x+4}}$$

$$\int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx$$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C_1$$

$$y + \ln|y+3|^{-5} = x + \ln|x+4|^{-5} + C_1$$

$$e^{LS} = e^{RS}$$

$$e^{y} \cdot e^{\ln|y+3|^{-5}} = e^{x} \cdot e^{\ln|x+4|^{-5}} \cdot e^{C_1}$$

$$\frac{e^{y}}{|y+3|^{5}} = \frac{e^{x} \cdot e^{C_1}}{|x+4|^{5}}$$

$$\frac{e^{y}}{(y+3)^{5}} = \pm e^{C_1} \cdot \frac{e^{x}}{(x+4)^{5}}$$

$$\frac{e^{y}}{(y+3)^{5}} = \frac{Ce^{x}}{(x+4)^{5}}$$

$$e^{y}(x+4)^{5} = Ce^{x}(y+3)^{5}$$

<u>Ex</u>: Find an implicit solution

$$(e^{3y} - ye^{y})\frac{dy}{dx} = e^{2y}\cos 5x, \quad y(0) = 7$$

$$(e^{y} - ye^{-y})\frac{dy}{dx} = \cos 5x$$

$$(e^{y} - ye^{-y})dy = \cos 5x \, dx$$

$$\int (e^{y} - ye^{-y})\,dy = \int \cos 5x\,dx$$

Integration by Parts for $\int -ye^{-y}\,dy$:

	D	I
$+$	$-y$	e^{-y}
$-$	-1	$-e^{-y}$
	0	e^{-y}

$$\int -ye^{-y}\,dy = ye^{-y} + e^{-y} + \text{constant}$$

$$e^{y} + ye^{-y} + e^{-y} = \frac{1}{5}\sin 5x + C_1$$

Sub $x=0$, $y=7$: $e^{7} + 7e^{-7} + e^{-7} = 0 + C_1$

$$C_1 = e^{7} + 8e^{-7}$$

$$e^{y} + (y+1)e^{-y} = \frac{1}{5}\sin 5x + e^{7} + 8e^{-7}$$

$$5e^{y} + 5(y+1)e^{-y} = \sin 5x + 5(e^{7} + 8e^{-7})$$

Ex: Find an explicit solution (get $y = \ldots$)
Interval of solution?

$$y' - ty^3 = 0 \quad , \quad y(0) = 2$$

$$\frac{dy}{dt} = ty^3$$

$$\frac{dy}{y^3} = t\,dt$$

$$-\frac{1}{2}y^{-2} = \frac{t^2}{2} + C_1$$

Sub $t=0$, $y=2$: $\quad -\frac{1}{2}\left(\frac{1}{4}\right) = 0 + C_1$

$$C_1 = -\frac{1}{8}$$

$$-\frac{1}{2}y^{-2} = \frac{t^2}{2} - \frac{1}{8}$$

Mult. by -8: $\quad 4y^{-2} = -4t^2 + 1$

$$\frac{4}{1-4t^2} = y^2$$

$$y = \pm\sqrt{\frac{4}{1-4t^2}}$$

or $y = \pm \dfrac{2}{\sqrt{1-4t^2}}$

Choose $+$ because $y(0) = 2$

$$\boxed{y = \frac{2}{\sqrt{1-4t^2}}}$$

Interval of Solution ?

$\{t \mid y \text{ is continuous}\}$

$1-4t^2>0$

$1>4t^2$

$\frac{1}{4}>t^2$

$\frac{1}{2}>|t|$

$\boxed{-\frac{1}{2}<t<\frac{1}{2}}$