1.1 Introduction

Differential equation (DE):
Equation involving derivatives

Ex: a) $\frac{dy}{dx} = 25 + y^2$

b) $x^2y'' - 2xy' + 6y = 0$

Ex: Confirm that $y = 5\tan(5x)$ is a solution to $\frac{dy}{dx} = 25 + y^2$

solution $\begin{cases} y = 5\tan(5x) \\ \frac{dy}{dx} = 25\sec^2(5x) \end{cases}$

$$\text{LS of DE} = \frac{dy}{dx}$$
$$= 25\sec^2(5x)$$

$$\text{RS} = 25 + y^2$$
$$= 25 + (5\tan(5x))^2$$
$$= 25 + 25\tan^2(5x)$$
$$= 25(1 + \tan^2(5x))$$
$$= 25\sec^2(5x)$$

LS = RS ✓

Ex. Let P: population t: time

Confirm that $P = \frac{6}{5}(1 - e^{-20t})$

Solves $\frac{dP}{dt} + 20P = 24$

$$\begin{cases} \boxed{P = \frac{6}{5}(1 - e^{-20t})} \\ \frac{dP}{dt} = \frac{6}{5}(20e^{-20t}) \\ \boxed{\frac{dP}{dt} = 24e^{-20t}} \end{cases}$$

$$\begin{aligned} \text{LS of DE} &= \frac{dP}{dt} + 20P \\ &= 24e^{-20t} + 20 \cdot \frac{6}{5}(1 - e^{-20t}) \\ &= 24e^{-20t} + 24 - 24e^{-20t} \\ &= 24 \\ &= \text{RS} \quad \checkmark \end{aligned}$$

Ex: Show that $x^2+y^2=25$ is a solution to $\frac{dy}{dx} = -\frac{x}{y}$

→ Find $\frac{dy}{dx}$ by <u>implicit differentiation</u>, then check.

Solution $x^2+y^2=25$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

LS of DE = RS ✓

<u>Notation for Derivatives:</u>

$y', y'', y''', y^{(4)}, y^{(5)}, \ldots$

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$

$\frac{dx}{dt}$ or $\dot{x}$

$\frac{d^2s}{dt^2}$ or $\ddot{s}$

} with respect to <u>time</u>

Order of a DE:

order of highest derivative

Ex: a) $y'' + 5(y')^3 + y = 0$ — 2nd order

b) $x^6 y''' - y = 0$ — 3rd order

c) $\frac{dy}{dx} = 25 + y^2$ — 1st order

Normal form of a DE:

Isolate highest derivative

Ex: $5xy'' + y = x$

$5xy'' = x - y$

$y'' = \frac{x-y}{5x}$

or

$y'' = \frac{-1}{5x} y + \frac{1}{5}$

} normal form

Linear DE: $y, y', y'', \ldots$ all to 1st power
Coefficients don't involve y

Ex: a) $3y'' + 2y = 0$

or

$3y'' + 0y' + 2y = 0$ LINEAR

b) $8y' + e^x y = 0$ LINEAR

c) $(\sin x)y' + 10y = 0$ LINEAR

d) $(\sin y)y' + 10y = 0$ NON-LINEAR

e) $y^2 = y''$ NON-LINEAR

Ex: Confirm that $y = C_1 \cos x + C_2 \sin x$
solves $y'' + y = 0$

C_1 and C_2 are parameters: any real numbers

$$\begin{cases} \boxed{y = C_1 \cos x + C_2 \sin x} \\ y' = -C_1 \sin x + C_2 \cos x \\ \boxed{y'' = -C_1 \cos x - C_2 \sin x} \end{cases}$$

LS of DE $= y'' + y = 0 =$ RS ✓

$y = C_1 \cos x + C_2 \sin x$ is a
"2-parameter family of solutions"

Some <u>particular</u> solutions:

$y = 0$

$y = 8\cos x$

$y = -4\sin x$

$y = \sqrt{2}\cos x + \pi \sin x$

etc.

<u>Ex</u>: Confirm that $x = Ct^4$ is a solution of $t\frac{dx}{dt} - 4x = 0$ (C is a parameter)

$$\begin{cases} x = Ct^4 \\ \frac{dx}{dt} = 4Ct^3 \end{cases}$$

$$\begin{aligned} \text{LS of DE} &= t\frac{dx}{dt} - 4x \\ &= t(4Ct^3) - 4(Ct^4) \\ &= 4Ct^4 - 4Ct^4 \\ &= 0 \\ &= \text{RS} \checkmark \end{aligned}$$

$x = Ct^4$ is a "1-parameter family of solutions"

Ex: Confirm that $y = \frac{1}{x}$ solves

$$xy' + y = 0$$

$$\begin{cases} y = x^{-1} \\ y' = -x^{-2} \end{cases}$$

LS of DE $= xy' + y$

$$= x(-x^{-2}) + x^{-1}$$

$$= -x^{-1} + x^{-1}$$

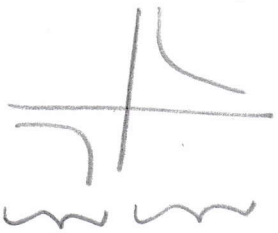
$$= 0$$

$$= RS \checkmark$$

Interval of solution:

Largest interval of x-values on which the solution y is continuous.
There may be several possible intervals.

Ex: Solution $y = \frac{1}{x}$



Continuous on $-\infty < x < 0$ and $0 < x < \infty$

Interval: $-\infty < x < 0$ OR $0 < x < \infty$