

1. [4 marks] Solve using the Laplace transform:
 $y'' - 10y' + 24y = 2e^{4t}$, $y(0) = 8$, $y'(0) = -3$

1) Apply \mathcal{L}

$$s^2 Y(s) - sy(0) - y'(0) - 10[sY(s) - y(0)] + 24Y(s) = \frac{2}{s-4}$$

$$s^2 Y(s) - 8s + 3 - 10[sY(s) - 8] + 24Y(s) = \frac{2}{s-4}$$

2) Solve for $Y(s)$

$$s^2 Y(s) - 8s + 3 - 10sY(s) + 80 + 24Y(s) = \frac{2}{s-4}$$

$$(s^2 - 10s + 24)Y(s) = 8s - 83 + \frac{2}{s-4}$$

$$Y(s) = \frac{8s - 83}{s^2 - 10s + 24} + \frac{2}{(s-4)(s^2 - 10s + 24)}$$

$$Y(s) = \frac{(8s - 83)(s-4) + 2}{(s-4)(s^2 - 10s + 24)}$$

$$Y(s) = \frac{8s^2 - 115s + 334}{(s-4)^2(s-6)}$$

3) Apply \mathcal{L}^{-1}

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{8s^2 - 115s + 334}{(s-4)^2(s-6)} \right\} \quad (*)$$

Partial Fractions

$$\frac{8s^2 - 115s + 334}{(s-4)^2(s-6)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{s-6} \quad \rightarrow$$

$$8s^2 - 115s + 334 = A(s-4)(s-6) + B(s-6) + C(s-4)^2$$

$$\text{Sub } s=4: \quad 2 = -2B \Rightarrow B = -1$$

$$s=6: \quad -68 = 4C \Rightarrow C = -17$$

$$s^2 \text{ coefficient:} \quad 8 = A + C \Rightarrow A = 25$$

from (*)

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{25}{s-4} - \frac{1}{(s-4)^2} - \frac{17}{s-6} \right\} \\ &= 25e^{4t} - te^{4t} - 17e^{6t} \end{aligned}$$

2. [5 marks] Solve using the Laplace transform:
 $y'' - 3y' = 10e^t \sin t, y(0) = 0, y'(0) = 0$

1) Apply \mathcal{L}

$$\mathcal{L}^2 Y(s) - s y(0) - y'(0) - 3[s Y(s) - y(0)] = \frac{10}{(s-1)^2 + 1}$$

$$s^2 Y(s) - 3s Y(s) = \frac{10}{s^2 - 2s + 2}$$

2) Solve for $Y(s)$

$$(s^2 - 3s) Y(s) = \frac{10}{s^2 - 2s + 2}$$

$$Y(s) = \frac{10}{s(s-3)(s^2 - 2s + 2)}$$

3) Apply \mathcal{L}^{-1}

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{10}{s(s-3)(s^2 - 2s + 2)} \right\} \quad (*)$$

Partial Fractions

$$\frac{10}{s(s-3)(s^2 - 2s + 2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C+D}{s^2 - 2s + 2}$$

$$10 = A(s-3)(s^2 - 2s + 2) + B s (s^2 - 2s + 2) + (C+D)s(s-3)$$

Sub $s=0$: $10 = -6A \Rightarrow A = -5/3$

Sub $s=3$: $10 = 15B \Rightarrow B = 2/3$

s^2 coefficient: $0 = A+B+C \Rightarrow C=1$

Sub $s=1$: $10 = -2A+B+(C+D)(-2)$

$$10 = -2A+B-2C-2D$$

$$D = -4 \quad \rightarrow$$

$$\text{From } (*) : y(t) = \mathcal{L}^{-1} \left\{ -\frac{5}{3} \cdot \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-4}{s^2-2s+2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ -\frac{5}{3} \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-4}{(s-1)^2+1} \right\}$$

Need $s-4$ in terms of $s-1$.

$$\text{Let } s-4 = a(s-1) + b$$

$$s-4 = s-1 + b$$

$$s-4 = (s-1) - 3$$

$$= \mathcal{L}^{-1} \left\{ -\frac{5}{3} \frac{1}{s} + \frac{2}{3} \frac{1}{s-3} + \frac{s-1}{(s-1)^2+1} - 3 \cdot \frac{1}{(s-1)^2+1} \right\}$$

$$= -\frac{5}{3} + \frac{2}{3} e^{3t} + e^t \cos t - 3e^t \sin t$$

3. [5 marks]

a) Solve using the Laplace transform:

$$y' + 8y = f(t), y(0) = 0, \text{ where } f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 9, & t \geq 3 \end{cases}$$

b) Find $y(1)$

c) Find $y(4)$

a)

1) Apply \mathcal{L}

$$\mathcal{L}Y(s) - y(0) + 8Y(s) = \mathcal{L}\{9U(t-3)\}$$

$$\mathcal{L}Y(s) + 8Y(s) = \frac{9e^{-3s}}{s}$$

$$\begin{aligned} e^{-3s} \mathcal{L}\{9\} \\ = \frac{e^{-3s} 9}{s} \end{aligned}$$

2) Solve for $Y(s)$

$$(s+8)Y(s) = \frac{9e^{-3s}}{s}$$

$$Y(s) = \frac{9e^{-3s}}{s(s+8)}$$

3) Apply \mathcal{L}^{-1}

$$y(t) = \mathcal{L}^{-1}\left\{\frac{9}{s(s+8)} e^{-3s}\right\} \quad (*)$$

Partial Fractions

$$\frac{9}{s(s+8)} = \frac{A}{s} + \frac{B}{s+8}$$

$$9 = A(s+8) + Bs$$

$$s=0: \quad 9 = 8A \Rightarrow A = 9/8$$

$$s=-8: \quad 9 = -8B \Rightarrow B = -9/8$$

From (*)

$$y(t) = \mathcal{L}^{-1}\left\{\left[\frac{9}{8} \frac{1}{s} - \frac{9}{8} \frac{1}{s+8}\right] e^{-3s}\right\} \rightarrow$$

$$y(t) = f(t-3)u(t-3)$$

$$F(s) = \frac{9}{8} \frac{1}{s} - \frac{9}{8} \frac{1}{s+8}$$

$$f(t) = \frac{9}{8} - \frac{9}{8} e^{-8t}$$

$$f(t-3) = \frac{9}{8} - \frac{9}{8} e^{-8(t-3)}$$

$$y(t) = \left[\frac{9}{8} - \frac{9}{8} e^{-8(t-3)} \right] u(t-3)$$

$$\text{or } \begin{cases} 0, & 0 \leq t < 3 \\ \frac{9}{8} - \frac{9}{8} e^{-8(t-3)}, & t \geq 3 \end{cases}$$

$$b) y(1) = 0$$

$$c) y(4) = \frac{9}{8} - \frac{9}{8} e^{-8}$$

4. [4 marks] Solve for $f(t)$:

$$f(t) = t^2 - \int_0^t f(\theta) e^{t-\theta} d\theta$$

$$f(t) = t^2 - f * e^t$$

Apply \mathcal{L} :

$$F(s) = \frac{2}{s^3} - F(s) \cdot \frac{1}{s-1}$$

Solve for $F(s)$:

$$F(s) + F(s) \frac{1}{s-1} = \frac{2}{s^3}$$

$$\left[1 + \frac{1}{s-1}\right] F(s) = \frac{2}{s^3}$$

$$\frac{s}{s-1} F(s) = \frac{2}{s^3}$$

$$F(s) = \frac{2(s-1)}{s^4}$$

Apply \mathcal{L}^{-1} :

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{2(s-1)}{s^4} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{2}{s^3} - \frac{2}{s^4} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{2!}{s^3} - \frac{1}{3} \frac{3!}{s^4} \right\}$$

$$= t^2 - \frac{t^3}{3}$$