

## 4.2 Determinants Cont'd

Ex: Solve using Cramer's Rule

$$\begin{cases} 2x + 3y + 2z = -11 \\ 3x \quad \quad + 5z = 23 \\ 4x + y + z = 1 \end{cases}$$

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 0 & 5 \\ 4 & 1 & 1 \end{vmatrix} = -3(1) - 5(-10) = 47$$

(2<sup>nd</sup> row)

$$\begin{bmatrix} + & - & \\ - & + & \dots \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \begin{matrix} -11 \\ 23 \\ 1 \end{matrix} & \begin{matrix} 3 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 2 \\ 5 \\ 1 \end{matrix} \end{bmatrix}$$

$\vec{b}$

$$|A_1| = \begin{vmatrix} -11 & 3 & 2 \\ 23 & 0 & 5 \\ 1 & 1 & 1 \end{vmatrix} = -23(1) - 5(-14) = 47$$

(2<sup>nd</sup> row)

$$|A_2| = \begin{vmatrix} 2 & -11 & 2 \\ 3 & 23 & 5 \\ 4 & 1 & 1 \end{vmatrix} = 2(18) + 11(-17) + 2(-89) = -329$$

$\vec{b}$   
(1<sup>st</sup> row)

$$|A_3| = \begin{vmatrix} 2 & 3 & -11 \\ 3 & 0 & 23 \\ 4 & 1 & 1 \end{vmatrix} = 188$$

$\uparrow$   
 $\vec{b}$

$$\begin{aligned} x &= \frac{|A_1|}{|A|} \\ &= \frac{47}{47} \\ &= 1 \end{aligned}$$

$$\begin{aligned} y &= \frac{|A_2|}{|A|} \\ &= \frac{-329}{47} \\ &= -7 \end{aligned}$$

$$\begin{aligned} z &= \frac{|A_3|}{|A|} \\ &= \frac{188}{47} \\ &= 4 \end{aligned}$$

Def

Cofactor: The signed determinant associated with a matrix entry

Ex:  $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$

Cofactor  $C_{11} = + \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} = -3$

$$\begin{bmatrix} + & - & \\ - & + & \dots \end{bmatrix}$$

$C_{12} = - \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -2$

$C_{32} = - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$

Cofactor Matrix =  $\begin{bmatrix} -3 & -2 & 1 \\ 11 & 4 & -3 \\ -6 & -2 & 2 \end{bmatrix}$

Def

Def

The adjoint of  $A$  : transpose of the cofactor matrix

Written  $\text{adj}(A)$

FACT

If  $A$  is  $n \times n$  and  $|A| \neq 0$  then

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

Ex: Find  $A^{-1}$  using the adjoint

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$$

$$|A| = -1(-11) - 3(3) = 2$$

(2<sup>nd</sup> row)

$$\begin{bmatrix} + & & \\ - & + & - \end{bmatrix}$$

$$\text{Cofactor Matrix} = \begin{bmatrix} -3 & -2 & 1 \\ 11 & 4 & -3 \\ -6 & -2 & 2 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} -3 & 11 & -6 \\ -2 & 4 & -2 \\ 1 & -3 & 2 \end{bmatrix}$$

transpose

$$A^{-1} = \frac{1}{2} \text{adj}(A)$$

Ex:  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Find  $A^{-1}$  using the adjoint

$$|A| = ad - bc$$

$$\text{Cofactor Matrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \leftarrow \text{transpose}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Review

$$\textcircled{1} \quad A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & 3 & 4 \\ 5 & 0 & 25 \end{bmatrix}$$

Find a basis for:

a)  $\text{Col}(A)$

b)  $\text{null}(A)$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 5R_1 \end{array} \quad \begin{bmatrix} 1 & -1 & 7 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$$

$$\frac{R_2}{5} \quad \begin{bmatrix} 1 & -1 & 7 \\ 0 & 1 & -2 \\ 0 & 5 & -10 \end{bmatrix}$$

$$\begin{array}{l} R_1 + R_2 \\ R_3 - 5R_2 \end{array} \quad \begin{bmatrix} \textcircled{1} & 0 & 5 \\ 0 & \textcircled{1} & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{RREF}$$

a) Use Columns 1 and 2 of A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

b)  $\text{null}(A) = \{ \vec{x} \text{ that solve } A\vec{x} = \vec{0} \}$

$$[A \mid \vec{0}]$$

$$\left[ \begin{array}{ccc|c} \textcircled{1} & 0 & 5 & 0 \\ 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\uparrow$$
$$\boxed{x_3 = t}$$

$$x_1 + 5x_3 = 0 \rightarrow \boxed{x_1 = -5t}$$

$$x_2 - 2x_3 = 0 \rightarrow \boxed{x_2 = 2t}$$

$$\vec{x} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} \right\}$$