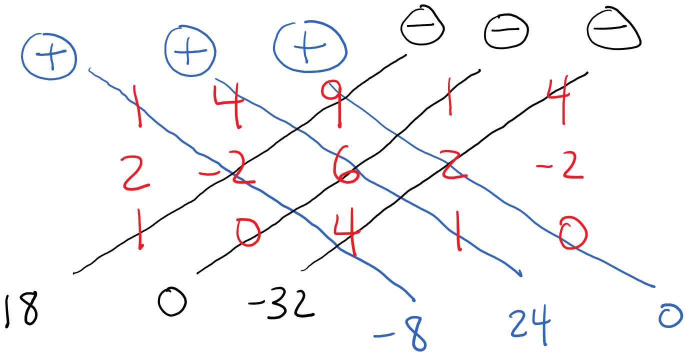
Week 9 Tuesday

March 3, 2020 8:38 AM

4.2 Determinants Cont'd

Quick Method for 3x3 Determinants

Ex: $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$ Find det A



$\det A = 18 - 32 - 8 + 24$
$= 2$

Fact

> If A is upper/lower triangular or diagonal then $|A|$ = product of diagonal entries

Ex: $\begin{vmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{vmatrix} = 2(-1)(4) = -8$

Why?

$\begin{vmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{vmatrix} = 4 \begin{vmatrix} 2 & 9 \\ 0 & -1 \end{vmatrix} = 4(2)(-1) = -8$

Row Operations to Calculate $|A|$

$R_i \leftrightarrow R_j$ changes sign of $|A|$

Can factor a row

$R_i \pm k R_j$ does not change $|A|$

Ex: Calculate $|A|$ using row operations

$$\begin{vmatrix} 1 & -2 & 1 & 9 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$$\boxed{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}}$$

$$= \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 5 & 1 & -15 \\ 0 & 7 & 1 & -22 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$$\boxed{R_2 \leftrightarrow R_4}$$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 7 & 1 & -22 \\ 0 & 5 & 1 & -15 \end{vmatrix}$$

$$\boxed{\begin{array}{l} R_3 \to R_3 - 7R_2 \\ R_4 \to R_4 - 5R_2 \end{array}}$$

$$= -\begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -6 & -64 \\ 0 & 0 & -4 & -45 \end{vmatrix} \qquad \boxed{\frac{R_3}{-6}}$$

$$= -(-6)\begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & -4 & -45 \end{vmatrix} \qquad \boxed{R_4 \to R_4 + 4R_3}$$

$$= 6\begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & 0 & \frac{-7}{3} \end{vmatrix} \qquad -45 + 4\left(\frac{32}{3}\right) = \frac{-7}{3}$$

Upper triangular

$$= 6\left(1 \cdot 1 \cdot 1 \cdot \frac{-7}{3}\right)$$

$$= -14$$

Fact

$$\boxed{\begin{array}{l}\text{An } n \times n \text{ matrix } A \text{ is invertible} \\ \quad \text{if and only if } \det A \neq 0\end{array}}$$

Properties of det A

① $\det A^{-1} = \dfrac{1}{\det A}$

② $\det(AB) = \det A \cdot \det B$

③ $\det(cA) = c^n \det A$ where A is $n \times n$

④ $\det(A^T) = \det A$

Ex: Use Property ② to show Property ①

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det I$$

$$\det A \cdot \det A^{-1} = 1$$

$$\det A^{-1} = \frac{1}{\det A}$$

Ex: Property ③

$$\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix} = 3\begin{vmatrix} a & b \\ 3c & 3d \end{vmatrix} = 3^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$|5A| = 5^4 |A|$ if A is 4×4

$|kA| = k^n |A|$ if A is $n \times n$

Cramer's Rule

Let A be $n \times n$

When $\det A \neq 0$, $A\vec{x} = \vec{b}$ has a unique solution:

$$i^{th} \text{ variable} = \frac{|A_i|}{|A|}$$

where $A_i = A$, with i^{th} column replaced by $\vec{b}$