Week 8 Friday

February 28, 2020 9:45 AM

4.1 Eigenvalues and Eigenvectors

Def

Let A be an $n \times n$ matrix.
If $A\vec{x} = \lambda\vec{x}$ for $\vec{x} \neq \vec{0}$ and for some number λ, then
λ is an eigenvalue of A and
$\vec{x}$ " eigenvector of A.

Ex: Show that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$
$$= 4\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= 4\vec{x}$$

Terminology: $\vec{x}$ is an eigenvector of A corresponding to eigenvalue $\lambda = 4$

Note: $A\vec{0} = \lambda\vec{0}$ is trivial
$\vec{0}$ is not considered an eigenvector

Ex: Find all eigenvectors of $A = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$ corresponding to $\lambda = 6$

Want
$$A\vec{x} = 6\vec{x}$$
$$A\vec{x} = 6I\vec{x}$$
$$A\vec{x} - 6I\vec{x} = \vec{0}$$
$$(A - 6I)\vec{x} = \vec{0}$$

$$[A - 6I \mid \vec{0}]$$

$$A - 6I = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -3 & -2 & 0 \\ -3 & -2 & 0 \end{array}\right]$$

$\frac{R_1}{-3}$ $$\left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ -3 & -2 & 0 \end{array}\right]$$

$R_2 + 3R_1$ $$\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = t$

$$x_1 + \frac{2}{3}x_2 = 0 \qquad x_1 = -\frac{2}{3}t$$

$$\vec{x} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} t \qquad (t \neq 0)$$

or $$\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} t \qquad (t \neq 0)$$

Check: $A\vec{x} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} -12 \\ 18 \end{bmatrix}$

$= 6\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ✓

Fact

To find eigenvectors of A corresponding to λ:

Solve $[A - \lambda I \mid \vec{0}]$

Def

The <u>eigenspace</u> E_λ is the set of all eigenvectors of A corresponding to λ, plus the zero vector.

It's a subspace of $\mathbb{R}^n$.

Ex: Find a basis for the eigenspace E_3 for $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 0 & 6 \\ 2 & 2 & -1 \end{bmatrix}$

Solve $[A - \lambda I \mid \vec{0}]$

$[A - 3I \mid \vec{0}]$

Subtract 3 from diagonal

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$

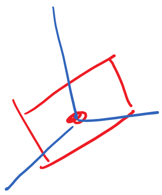
$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_2 = s$ $\quad$ $x_3 = t$

$x_1 + x_2 - 2x_3 = 0$ $\qquad$ $x_1 = -s + 2t$

List of eigenvectors $\qquad \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$

Basis for $E_3 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

E_3 

<u>Ex</u>: Find a basis for E_0 for $A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

Solve $[A - \lambda I \mid \vec{0}]$

$[A \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 4 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_2 = t$

$$\vec{x} = \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} t \qquad (t \neq 0)$$

Basis for $E_0 = \left\{ \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \right\}$

Applications

1. Vibrations

eigenvalues = natural frequencies

eigenvectors = shapes/directions of vibrations

2. Facial Recognition (Image Processing)

$$[\text{Image}] \rightarrow \begin{bmatrix} 0 & 18 & 256 \\ & \cdots & \end{bmatrix}$$

eigenvectors = "basic" faces

Goal: Express any face as a linear combination of basic faces