3.6 Linear Transformations

<u>Ex</u>: Transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$

Rotates each vector by $90°$ counterclockwise.

<u>Notation</u>: $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

<u>Terminology</u>: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is the <u>image</u> of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under T

Def

> The <u>matrix transformation</u> T_A multiplies a vector on the left by A.

$$T_A(\vec{v}) = A\vec{v}$$

<u>Ex</u>: a) $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

Find $T_A\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

$= A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x + z \\ -x + y - 3z \end{bmatrix}$$

b) $T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$

Find A

$$\begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ \\ \\ \end{bmatrix}$$

Def

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if :

1) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

2) $T(c\vec{u}) = cT(\vec{u})$ for all constants c

Ex: Show that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 + x \end{bmatrix}$ is not linear

- Show that 1 property fails
- Can use numbers

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix}2\\0\end{bmatrix} \neq 2\,T\begin{bmatrix}1\\0\end{bmatrix}$$

Ex: Confirm that T_A satisfies Property 2)
where $A = \begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 1\end{bmatrix}$

$$T_A(c\vec{u}) = \begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}cu_1\\ cu_2\end{bmatrix}$$

$$= \begin{bmatrix}cu_1 + cu_2\\ cu_1 + 2cu_2\\ cu_1 + cu_2\end{bmatrix}$$

$$= c\begin{bmatrix}u_1 + u_2\\ u_1 + 2u_2\\ u_1 + u_2\end{bmatrix}$$

$$= c\,T_A(\vec{u})$$

2 properties

FACT

T is a linear transformation if and only if T is a matrix transformation

T_A

- Tells us which transformations are linear

Def

The standard matrix for T is the matrix that performs T.

Notation: $[T]$

How to find $[T]$:

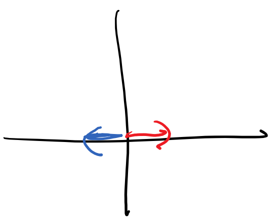
$$[T] = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} \quad T\begin{bmatrix}0\\0\\1\end{bmatrix}$$

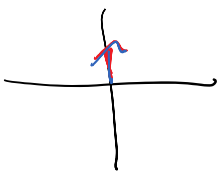
Works because T is <u>linear</u>

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$
Reflects a vector in the y-axis

a) Find $[T]$



$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$



$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix}1\\0\end{bmatrix} \quad T\begin{bmatrix}0\\1\end{bmatrix}$$

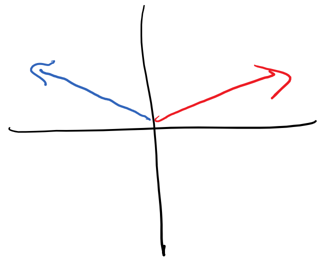
b) Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

3) Find $T\begin{bmatrix} x \\ y \end{bmatrix}$

$$= [T]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

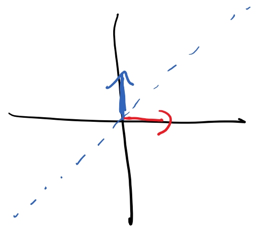
$$= \begin{bmatrix} -x \\ y \end{bmatrix}$$



Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

Reflects a vector in $y = x$

Find $[T]$



$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$