3.3 The Inverse of a Matrix Cont'd

Ex: Find A^{-2} for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-n} = (A^n)^{-1} \text{ or } (A^{-1})^n$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$A^{-2} = \frac{1}{4}\begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$$

Ex: A, B, X are all $n \times n$ invertible matrices.

Given $(AX)^{-1} = BA$

Solve for X

$$((AX)^{-1})^{-1} = (BA)^{-1}$$

$$AX = A^{-1}B^{-1}$$

socks and shoes

Multiply by A^{-1} on the left:

$$\underline{\underline{A^{-1}}}AX = \underline{\underline{A^{-1}}}A^{-1}B^{-1}$$

$$X = A^{-2}B^{-1}$$

$$\boxed{\begin{aligned} C^{-1}C &= I \\ CC^{-1} &= I \\ IX &= X \end{aligned}}$$

Elementary Matrices: represent row operations

How is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing?

Ex: a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ represents $3R_1$ [ ]

$\frac{R_1}{3}$ [ ] undoes it

$$E_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$

$R_1 \leftrightarrow R_2$ undoes it $\qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represents $R_2 + 3R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

$R_2 - 3R_1 \begin{bmatrix} & \\ & \end{bmatrix}$ undoes it

$$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

FACT Elementary matrices act on the left of A

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$ $\qquad \cancel{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & \\ 2c & \end{bmatrix}}$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$R_1 - \frac{1}{2}R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

How is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing?

$$\underbrace{E_2 E_1}\, A = I$$

$$\boxed{A^{-1} = E_2 E_1}$$

$$A = (A^{-1})^{-1}$$

$$A = (E_2 E_1)^{-1}$$

$$A = E_1^{-1} E_2^{-1}$$

$$\boxed{A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$$

$2R_1 \begin{bmatrix} & \\ & \end{bmatrix} \qquad R_1 + \frac{1}{2}R_2 \begin{bmatrix} & \\ & \end{bmatrix}$

Ex: $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

Write A and A^{-1} as a product

of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 - R_1$ $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\frac{R_2}{-1}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

How is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing ?

$$\underbrace{E_4 E_3 E_2 E_1}\, A = I$$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$A = (E_4 E_3 E_2 E_1)^{-1}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$2R_1$ $\begin{bmatrix} & \\ & \end{bmatrix}$

$R_2 + R_1$ $\begin{bmatrix} & \\ & \end{bmatrix}$